Ministry of Higher Education and Scientific Research
Daiyla University – College of Engineering

Mathematics I

Mathematics I

For

First Stages Engineering Students

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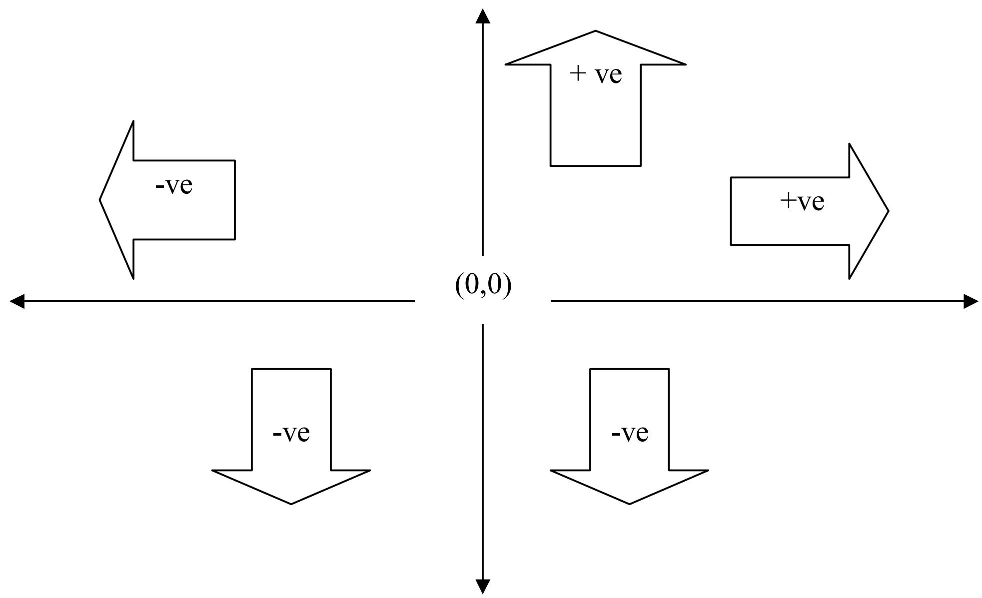
Contents

Chapter One

Basic Principles and Review

1.1 Cartesian Coordinates:

Cartesian coordinate, make it possible to graph algebraic equation in two variables, such as lines and curves. The also allow us to calculate angles and distance, and to write coordinate equations to describe the paths along which objects move. The horizontal line is called the x-axis and the vertical line is the y-axis. The point where the line cross is the origin.

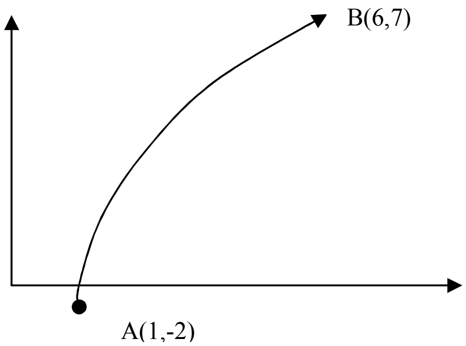


1.2 Slope of the line:

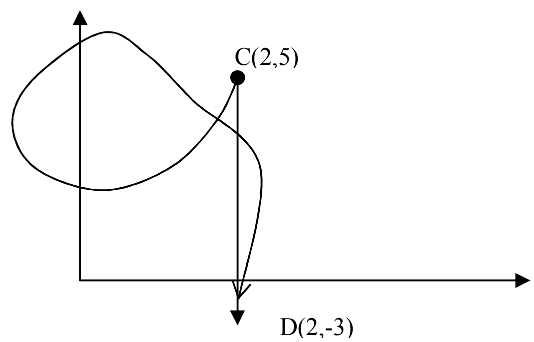
When a particle moves from one position in the plane to another, the net changes in the particle's coordinates are calculated by subtracting the coordinate of starting point from the coordinates of stopping point.

For example 1, as shown in figure, the path of particle that moved from A to B. the net change in the x-coordinate was, $\Delta x=6-1=5$.

The y-coordinate increased from $y=-2$ to $y=7$, so $\Delta y=7-(-2)=9$.



Example 1



Example 2

Another example 2, the net change in coordinates in moving from C to D along either path,

$\Delta x=2-2=0$ and $\Delta y=-3-5=-8$. Note that values of Δx and Δy do not depend on the path taken.

1.2.1 Slopes of non vertical lines:

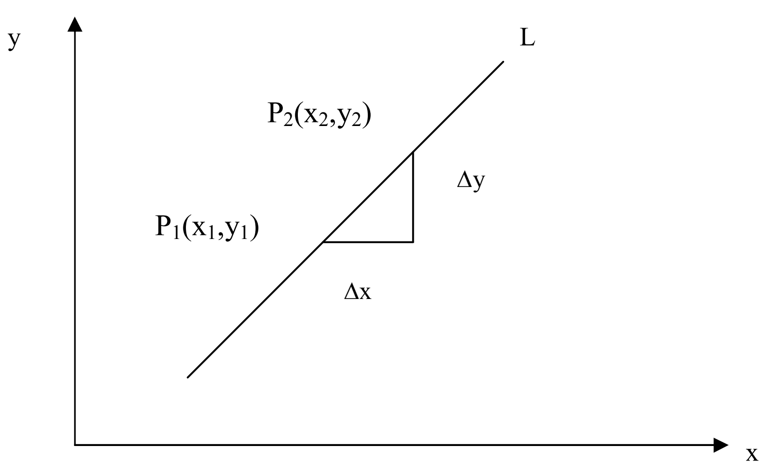
All lines expect the vertical lines have slopes. If L is non vertical line:

$\Delta y = y_2-y_1$=rise from P_1 to P_2

$\Delta x= x_2-x_1$=run from P_1 to P_2

Since L is non vertical, so that, $\Delta x \neq 0$

$\therefore slope = \dfrac{\Delta y}{\Delta x}$, the amount of rise per unit of run.



The horizontal line has slope zero, since $\Delta y=0$.

The formula $m = slope = \dfrac{\Delta y}{\Delta x}$ does not apply to vertical lines because $\Delta x=0$, so the slope of vertical lines is undefined because, $m = \dfrac{\Delta y}{0} = \infty$.

Example: Calculate the slope of the line through the point $P_1(1,2)$ and $P_2(2,5)$.

Solution:

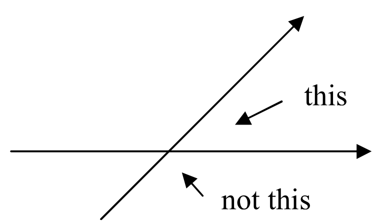
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{2-1} = \frac{3}{1} = 3$$

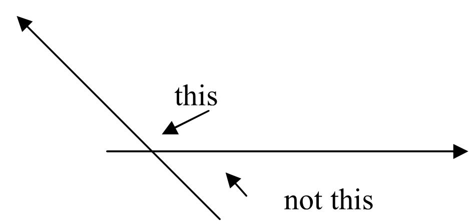
m=3 means that y increase 3 units every time x increase 1 units, this mean that change in y is 3 times change in x ($\Delta y=3\Delta x$).

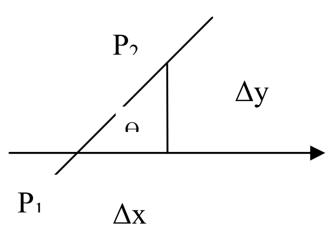
1.2.2 Angels of Inclination:

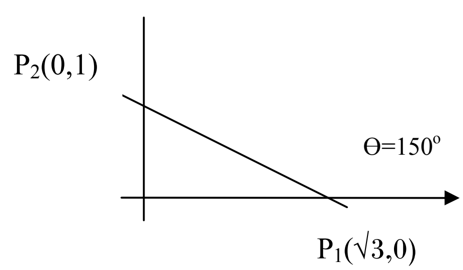
The angels of inclination of a line crosses the x-axis is the smallest angle we get when we measure counterclockwise from the x-axis around the point of intersection. The angle of inclination of horizontal line is taken to be *zero*. Thus, angles of inclination may have any measure from 0^o up to but not include 180^o.

Slope = m = tan θ , how?





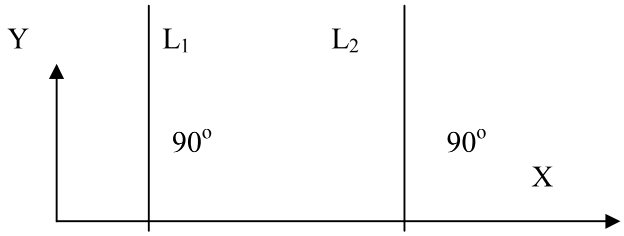




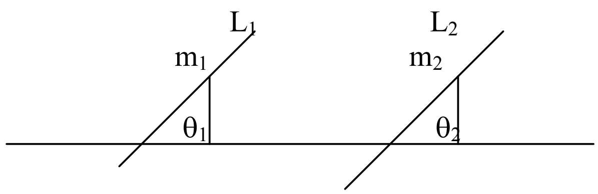
$m = slope = \dfrac{\Delta y}{\Delta x} = \tan\theta$, $\quad m = \dfrac{\Delta y}{\Delta x} = \dfrac{-1}{\sqrt{3}} = -0.577$ or, $\tan\theta=\tan 150^o=-0.577$

1.2.3 Parallel and Perpendicular Lines:

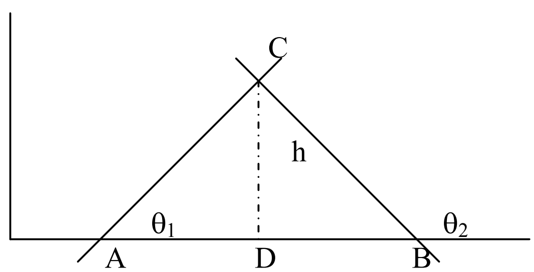
Vertical lines have no slopes.



Parallel lines have equal angles of inclination; if they are not vertical they have the same slopes. Conversely, lines with equal slopes have equal angles of inclination and therefore parallel.



If neither of tow perpendicular lines L_1 and L_2 is vertical, their slopes m_1 and m_2 are related by $m_1m_2=-1$.



Example: L is perpendicular to line whose slope is (-3/4). Find slope of L?

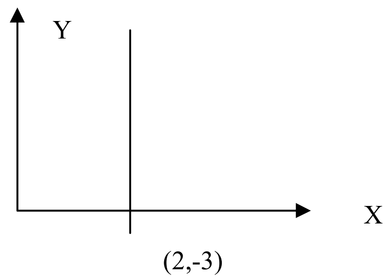
Solution:
Since, $m_1=-3/4$, $m_1m_2=-1$, therefore, $m_2=4/3$.

1.3 Equation of Lines:

An equation of line is an equation that satisfied by coordinate of the points that lies on the line and is not satisfied by the coordinates of the point that lies else where. They tell us when lines are vertical and what their slopes are when they are not vertical.

- Vertical line:- the standard equation for vertical line through point (a,b) is x=a.

For example, standard equation of vertical line through point (2,-3) is x=2



- Non vertical line: if $P_1(x_1,y_1)$ and P(x,y) is another point, then:

$$m=\frac{y-y_1}{x-x_1} \quad , \quad \therefore y-y_1=m(x-x_1)$$

This called *point-slope* equation.

Example: write the equation for the line through point (1,2) and slope=-3/4.

Solution:

$$\therefore y-y_1=m(x-x_1) \quad , \quad \therefore y-2=-\frac{3}{4}(x-1)$$

And, $y=-\frac{3}{4}x+\frac{11}{4}$.

- Intercepts: the points when a line crosses the x-axis is x-intercept of the line. To find it, set y=0 in equation of line and solve for x.
- The point when line crosses the y-axis is y-intercept. To find it, set x=0 and solve for y.

Example: for the line equation y=2x-3

Solution:

Let y=0, the x-intercept is x=3/2

Let x=0, the y-intercept is y=-3.

Therefore, Slope-intercept equation is y=mx+b.

Example: write the line equation, if the slope=-3/4 and intercept=5

Solution:

$$y=-\frac{3}{4}x+5$$

Example: find the slope and y-intercept for the line 8x+5y=20.

Solution:

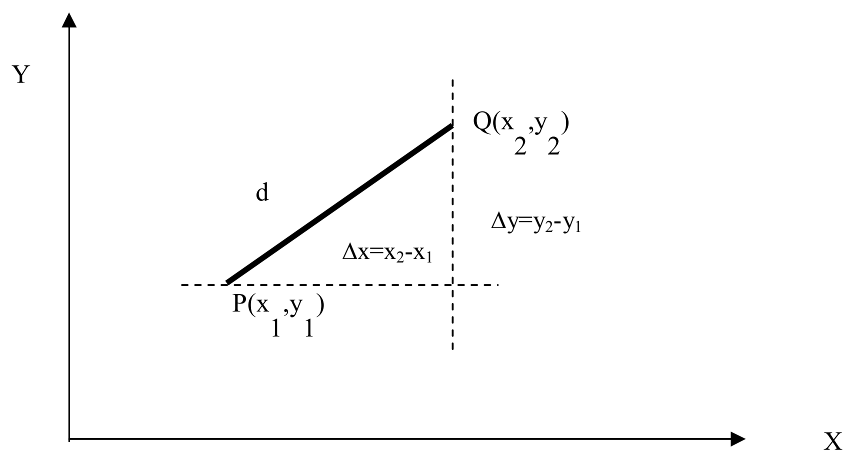
$$5y = -8x + 20.$$

$$y = -\frac{8}{5}x + 4$$

So, m=-8/5 and, b=4.

- Horizontal lines: for horizontal line (m=0) and y=mx+b, therefore, y=b.
- The distance from a point to a line: to calculate the distance between the point $P(x_1,y_1)$ and $Q(x_2,y_2)$, we use the following formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example: Find the distance from point P(2,1) to the line L, y=x+2.

Solution:

First find equation of line L\` normal on L, through the point P:

$$\therefore y - y_1 = m(x - x_1) \quad \text{,and} \quad m_1 m_2 = -1$$

So, $m_2=-1$, and $y-1=-1(x-2)$

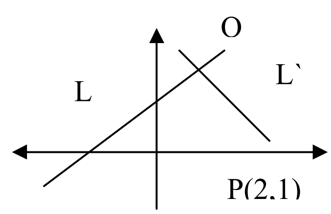
Equation of normal is; $y= -x+3$

Second find the point Q by solving equation of line L with equation of line L\`;

x+2=-x+3 $\Longrightarrow$ x=1/2 and y=5/2

So, the point Q(1/2,5/2)

Third, find the distance between P and Q:



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(2 - \frac{1}{2})^2 + (1 - \frac{5}{2})^2} = \sqrt{\frac{18}{4}} = \frac{3}{2}\sqrt{2} \quad \text{Unit length.}$$

1.4 Functions and Graph:

Function is the relation between the independent variable x, and dependent variable y, so that, y=f(x).

Examples: find the Domain and Range of following functions:

$y=x^2$ domain $-\infty \le x \le \infty$, and range $y \ge 0$

$y = \sqrt{1 - x^2}$ domain $-1 \le x \le 1$, and range $0 \le y \le 1$

$y = \frac{1}{x}$ domain $x \neq 0$ and range $y \neq 0$

$y = \sqrt{x}$ domain $x \ge 0$ and range $y \ge 0$

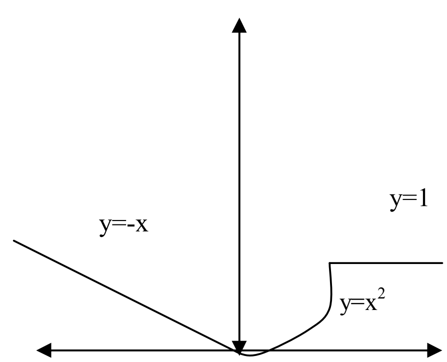
$y = \sqrt{4 - x}$ domain $x \le 4$ and range $y \ge 0$

1.4.1 Functions Defined in Pieces:

The functions graphed so far have been defined over their domains by single formulas. Some functions are defined by applying different formulas to different part of their domains.

Example:

$$y = f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$



H.W:

1. Use slopes to determine in each case wither the points are lie on a common straight line:

a. A(1,0), B(0,1), C(2,1)

b. A(-2,1), B(0,5), C(-1,2)

c. A(-3,-2), B(-2,0), C(-1,2), D(1,6)

2. In problem below find the slopes and intercept:

a. x+y=2, b. x-2y=4, c. 2y=3x+5, d. 3x+4y=12

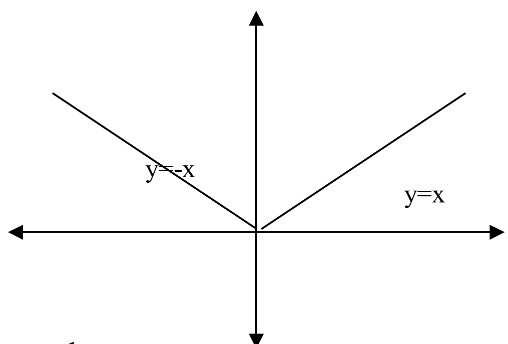
3. Find the domain and range of the function and graph the function:

a. $y=x^2+1$, b. $y=-x^2$, c. y=-(1/x), d. y=(2x)

1.5 Absolute Values:

The absolute value of x:

$$|x| = \sqrt{x^2} = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x \leq 0 \end{cases}$$



The function y=|x| is graphed as shown above.

Examples:

|3|=3 , |0|=0, |-5|=5

Example: solve |2x-3|=7

2x-3=±7 2x=3±7

Either 2x=10 or 2x=-4

So, x=5 or x=-2.

* The absolute value of product of two number is the product of their absolute values, |ab|=|a||b|

Examples:

$|(-1)(4)|=|-1|\,|4|=(1)(4)=4$

$|3x|=|3|\,|x|=3|x|$

$|-2(x+5)|=|-2|\,|x+5|=2|x+5|$

* The absolute value of a sum of two numbers is never larger than the sum of their absolute value, $|a+b|\leq|a|+|b|$

Examples:

$|0+5|=5 \leq |0|+|5|=0+5=5$

$|-3+0|=3 \leq |-3|+|0|=3+0=3$

$|3+5|=8 \leq |3|+|5|=3+5=5$

* $|a-b|=|b-a|$ because $|a-b|=|(-1)(b-a)|= |-1|\,|b-a|=|b-a|$

* $|a|< c \quad \Leftrightarrow \quad -c < a <c$

The inequality $|a|< 5$ says that the distance from (a) to the origin is less than 5 units. This equivalent to saying that (a) lies between -5 and 5. In symbols:

$$|a|< 5 \Leftrightarrow \quad -5 < a <5$$

Example: Find the values of x that satisfy the inequality $|x-5|< 9$.

$$\begin{aligned}&|x-5|<9\\&-9<x-5<9\\&-9+5<x-5+5<9+5\\&-4<x<14\end{aligned}$$

Example: Find the values of x that satisfy the inequality

$|(3x+1)/2|< 1$.

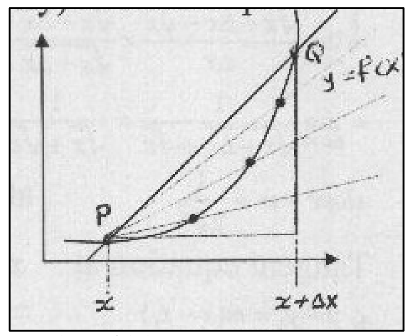
$$\begin{aligned}&-1<\frac{3x+1}{2}<1\\&-2<3x+1<2\\&-3<3x<1\\&-1<x<\frac{1}{3}\end{aligned}$$

H.W

1. $|2x+4| < 1$
2. $|(2x+1)/3| < 1$
3. $|x| < 2$
4. $|x-1| \leq 2$
5. $|x+1| < 3$
6. $|2x+2| < 9$
7. $|1-2x| \leq 9$

1.6 Slope of Curve:

The derivative of the function, let P(x,y) be a point on the graph of the function y=f(x), if Q(x+Δx, y+Δy) is another point on the graph, then;



y+Δy=f(x+Δx), subtract *y=f(x)*

Then, *Δy= f(x+Δx)-f(x)*

the slope of the line PQ is:

$$slope = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Slope of the curve at P=lim(slope of PQ)= $\lim \frac{\Delta y}{\Delta x} = \lim \frac{f(x+\Delta x) - f(x)}{\Delta x}$

So, the derivative of function (f \`prime);

$$f`(x) = \lim \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- A function that has derivative at a point x is said to be differentiable at x.
- A function that is differentiable at every point of its domain is said to be differentiable.
- The most common notation for derivative f \`(x), y\`, dy/dx, and df/dx

Example: find the derivative and slope at x=3 and write equation of tangent? For $f(x) = x^2, and, f(x) = 1 + \sqrt{x}$

1- For $f(x) = x^2$

$$f`(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+\Delta x)\Delta x}{\Delta x} = 2x$$

Then slope, m = 2 * 3 =6 and y=9

Tangent equation is, $\therefore y - y_1 = m(x - x_1)$

6x-y=9 $\Rightarrow$ y=6x-9

2- For $f(x) = 1 + \sqrt{x}$

$$f`(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1 + \sqrt{x+\Delta x} - (1+\sqrt{x})}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \times \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} = \lim_{\Delta x \to 0} \frac{x + \Delta x + \sqrt{x}\sqrt{x+\Delta x} - \sqrt{x}\sqrt{x+\Delta x} - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$slope = m = \dfrac{1}{2\sqrt{3}}$ at x=3

Tangent equation at $x_1=3$ $y_1=1+\sqrt{3}$

$\therefore y - y_1 = m(x - x_1)$ $\Rightarrow$ $x - 2\sqrt{3}y = -(3 + 2\sqrt{3})$

1.7 Velocity:

In the interval from time (t) to time (t+Δt) the body moves from position s=f(t) to position s+Δs=f(t+Δt), for net change or displacement, Δs=f(t+Δt)-f(t).

$$V_{av} = \frac{displacement}{time} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

The instantaneous velocity at $\Delta t \to 0$:

$$V = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{ds}{dt} = f`(t)$$

Example: a body falls with s=(1/2)gt^2, find instantaneous velocity as a function of t. how fast is the body falling in feet per seconds, 2 seconds after release?

Solution:

$$V = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\frac{1}{2}g(t+\Delta t)^2 - \frac{1}{2}t^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{1}{2}g\left(t^2 + 2t\Delta t + \Delta t^2 - t^2\right)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{2}g(2t + \Delta t) = \frac{1}{2} \times 2gt = gt$$

Then, $V=gt$

The acceleration, in English unit g=32 ft/sec^2, and in SI unit g=9.81 m/sec^2,

Then, v=(32)(2)=64 ft/sec

H.W: find the displacement and average velocity for time interval from, t=0 to t=2 seconds for,

- $S=2t^2+5t-3$ *answer*: $\Delta s=18$ $V_{av}=9$
- $S=4-2t-t^2$ $\Delta s=-8$ $V_{av}=-4$

  $S=4t+3$ $\Delta s=8$ $V_{av}=4$

1.8 ***Limits:***

The limits combination thermo, if:

$$\lim_{t\to c} f_1(t)=L_1, and, \lim_{t\to c} f_2(t)=L_2, then:$$

$$1.\lim_{t\to c}[f_1(t)+f_2(t)]=L_1+L_2$$

$$2.\lim_{t\to c}[f_1(t)-f_2(t)]=L_1-L_2$$

$$3.\lim_{t\to c}[f_1(t).f_2(t)]=L_1.L_2$$

$$4.\lim_{t\to c}\left[\frac{f_1(t)}{f_2(t)}\right]=\frac{L_1}{L_2}\Rightarrow if, L_2\neq 0$$

L_1 and L_2 are real number.

5. Limits of polynomials, if:

$$f(t)=a_n t^n + a_{n-1}t^{n-1}+...+a_o$$

$$Then:$$

$$\lim_{t\to c} f(t)=f(c)=a_n c^n + a_{n-1}c^{n-1}+...+a_o$$

Example: find $\lim_{t\to 2}\frac{t^2+2t+4}{t+2}$ and, $\lim_{t\to 2}\frac{t^3-8}{t^2-4}$

$$\lim_{t\to 2}\frac{t^2+2t+4}{t+2}=\frac{2^2+2\times 2+4}{2+2}=\frac{12}{4}=3$$

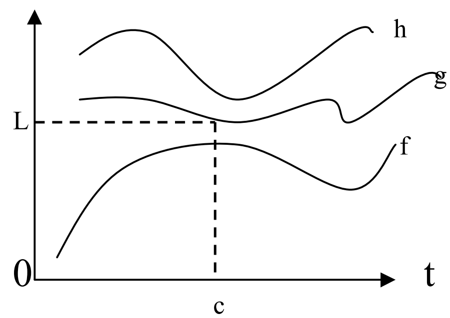
$$\lim_{t\to 2}\frac{t^3-8}{t^2-4}=\lim_{t\to 2}\frac{(t-2)(t^2+2t+4)}{(t-2)(t+2)}=\lim_{t\to 2}\frac{t^2+2t+4}{t+2}=\frac{2^2+2\times 2+4}{2+2}=\frac{12}{4}=3$$

Note the difference between the direct substitution in the first, and the simplification in the second.

1.8.1 The Sandwich Theorem:

Suppose that, f(t)≤ g(t)≤ h(t) for all t≠c in some interval about c and that f(t) and h(t) approach the same limit, L as t→c, then,

$$\lim_{t\to c} g(t) = L\,.$$



Application of Sandwich Theorem,

1. $\lim_{\theta\to 0}\sin\theta = 0$
2. $\lim_{\theta\to 0}\cos\theta = 1$
3. $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Example: $\lim_{x\to 0}\frac{\sin 3x}{x}$

$$\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{3x\to 0}\frac{3\sin 3x}{3x} = \lim_{\theta\to 0}\frac{3\sin\theta}{\theta} = 3\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 3\times 1 = 3$$

Example:

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\times\frac{1}{\cos x}\right) = \left[\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\lim_{x\to 0}\left(\frac{1}{\cos x}\right)\right] = (1)(1) = 1$$

1.8.2 Right Hand Limits and Left Hand Limits:

Some times the values of a function f(t) tend to deferent limits as it approaches a number c from different sides. When this happens, we call the limit of F as t approach c from the right, the Right-hand limit of F at c, and the limit of F at t approach c from the left the Left- hand limit.

1.8.3 The Greatest Integer Function:

[x] is called the greatest integer in x.

Positive values [1.9]=1

	[2]=2
	[3.4]=3
The zero value	[0.5]=0
	[0]=0
Negative value	[-1.2]=-2
	[-0.5]=-1

Example: $\lim_{x \to 1}\left[\frac{x}{2}\right]$

Right-hand: $\lim_{x \to 1^+}\left[\frac{1.1}{2}\right] = [0.55] = 0$

Left-hand $\lim_{x \to 1^-}\left[\frac{0.9}{2}\right] = [0.45] = 0$

So, Right-hand limit = Left –hand limit.

Example: $\lim_{x \to 2}\left[\frac{x}{2}\right]$

Right-hand: $\lim_{x \to 2^+}\left[\frac{2.1}{2}\right] = [1.05] = 1$

Left-hand $\lim_{x \to 2^-}\left[\frac{1.9}{2}\right] = [0.95] = 0$

So, Right-hand limit ≠ Left –hand limit.

Example: $\lim_{x \to 3}[2x]$

R: $\lim_{x \to 3^+}[2 \times 3.1] = [6.1] = 6$

L: $\lim_{x \to 3^-}[2 \times 2.9] = [5.8] = 5 \Rightarrow \therefore R \neq L$

Example: $\lim_{x \to 2}\frac{[x]}{x} = \frac{\lim_{x \to 2}[x]}{\lim_{x \to 2} x}$

R: $\frac{\lim_{X \to 2^+}[2.1]}{\lim_{X \to 2^+} 2} = \frac{2}{2} = 1$

L: $\frac{\lim_{X \to 2^-}[1.9]}{l\ im_{X \to 2^-}\ 2} = \frac{1}{2} = 0.5 \Rightarrow \therefore R \neq L$

Hint: In computer science, the common notation for greatest integer in x is, $\lfloor x \rfloor$ (integer floor), it suggest an integer floor of x. The notation $\lceil x \rceil$, (integer ceiling), is used for the smallest integer greater than or equal to x.

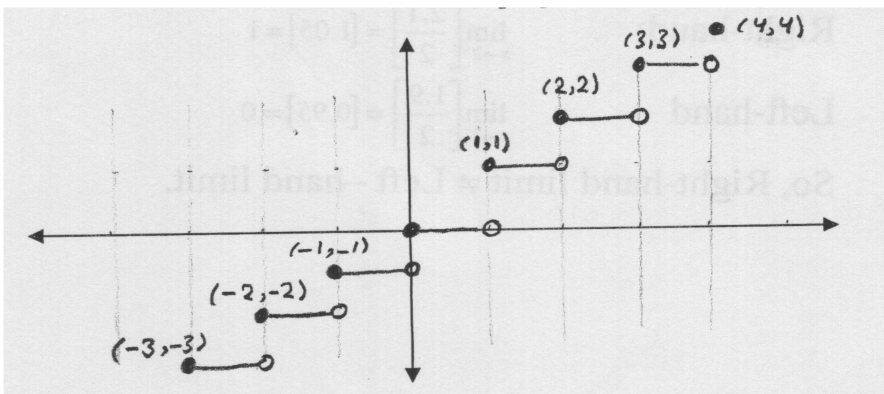
The greatest integer function is a *Step Function*; step function exhibits points of discontinuity. Where they jumped from one value to another without taking on any of intermediate values.

Example: Graph $y = [x]$

y=0 0<x<0.99 this mean, $[0] = 0, [0.5] = 0, [0.9] = 0$

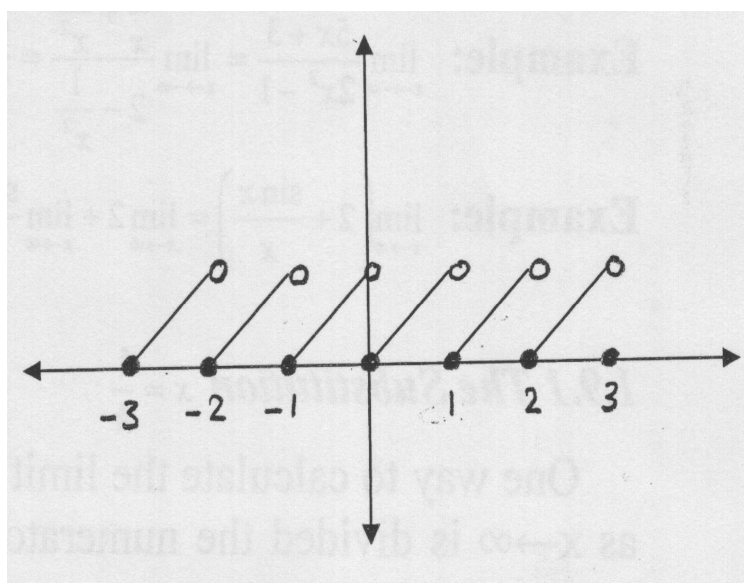
y=1 1<x<1.99 this mean, $[1] = 1, [1.5] = 1, [1.9] = 1$

y=-1 -1<x<-0.00..this mean, $[-1] = -1, [-0.5] = -1, [-0.005] = -1$



Example: Graph the function $y = x - [x]$, $-3 \leq x \leq 3$

X	$y = x - [x]$
0	0
0.1	$0.1 - [0.1] = 0.1 - 0 = 0.1$
0.5	$0.5 - [0.5] = 0.5$
1	1-1=0
1.5	$1.5 - [1.5] = 1.5 - 1 = 0.5$
1.9	0.9
2	0



H.W:

1. Graph $y = \left[\dfrac{x}{2}\right] \Rightarrow given -3 \le x \le 3$
2. Graph $y = [2x] - 2[x] \Rightarrow given -3 \le x \le 3$

1.9 Infinity as Limits:

The combination theorem for limits at infinity, if

$\lim\limits_{x\to\infty} f(x) = L_1, and \lim\limits_{x\to\infty} g(x) = L_2$ Where, L_1 and L_2 are real numbers.

1. $\lim\limits_{x\to\infty}[f(x) + g(x)] = L_1 + L_2$
2. $\lim\limits_{x\to\infty}[f(x) - g(x)] = L_1 - L_2$
3. $\lim\limits_{x\to\infty}[f(x).g(x)] = L_1.L_2$
4. $\lim\limits_{x\to\infty} kf(x) = kL_1$ k, is any number.
5. $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$ $L_2 \neq 0$

Example: $\lim\limits_{x\to\infty} \dfrac{x}{7x+4}$

Solution: divided by greatest power in dominator.

$$\lim_{x\to\infty} \frac{x}{7x+4} = \lim_{x\to\infty} \frac{1}{7+\frac{4}{x}} = \frac{1}{7+0} = \frac{1}{7}$$

Example: $\lim\limits_{x\to\infty} \dfrac{2x^2 - x + 3}{3x^2 + 5}$.

Solution: $$\lim_{x\to\infty} \frac{2x^2 - x + 3}{3x^2 + 5} = \lim_{x\to\infty} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{3 + \frac{5}{x^2}} = \frac{2-0+0}{3+0} = \frac{2}{3}.$$

Example: $$\lim_{x\to-\infty} \frac{5x+3}{2x^2-1} = \lim_{x\to-\infty} \frac{\frac{5}{x} + \frac{3}{x^2}}{2 - \frac{1}{x^2}} = \frac{0+0}{2-0} = 0.$$

Example: $$\lim_{x\to\infty}\left(2 + \frac{\sin x}{x}\right) = \lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{\sin x}{x} = 2 + 0 = 2.$$

1.9.1 The Substitution $x = \frac{1}{h}$

One way to calculate the limit of quotient of two polynomials as x→∞ is divided the numerator and dominator by the largest power of x in dominator. Another way is to let $x = \frac{1}{h}$, and calculate the limit as $h \to 0^+$.

Example: $\lim\limits_{x\to\infty} \frac{2x^2 - x + 3}{3x^2 + 5}$. Use the substitution $x = \frac{1}{h}$

Solution: $$\lim_{h\to 0^+} \frac{\frac{2}{h^2} - \frac{1}{h} + 3}{\frac{3}{h^2} + 5} = \lim_{h\to 0^+} \frac{2 - h + 3h^2}{3 + 5h^2} = \frac{2}{3}.$$

Example: $$\lim_{x\to -\infty} \frac{5x+3}{2x^2 - 1} = \lim_{h\to 0^+} \frac{\frac{5}{h} + 3}{\frac{2}{h^2} - 1} = \lim \frac{5h + 3h^2}{2 - h^2} = \frac{0}{0} = 0.$$

To calculate the limit as x→ -∞, we substitute $x = \frac{1}{h}$, and calculate the limit as $h \to 0^-$.

Example: $$\lim_{x\to \infty^-} \frac{2x^2 - 3}{7x + 4} = \lim_{h\to 0^-} \frac{\frac{2}{h^2} - 3}{\frac{7}{h} + 4} = \lim_{h\to 0^-} \frac{2 - 3h^2}{7h + 4h^2} = -\infty$$

Examples:

1. $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$
2. $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$
3. $\lim\limits_{x\to 0^+} \frac{1}{x^2} = \infty$
4. $\lim\limits_{x\to 1^-} \frac{1}{x-1} = -\infty$
5. $\lim\limits_{x\to 2^-} \frac{1}{x^2 - 4} = -\infty$
6. $\lim\limits_{x\to 2^+} \frac{1}{x^2 - 4} = +\infty$
7. $\lim\limits_{x\to \infty} \sqrt{x} = \infty$
8. $\lim\limits_{x\to -\infty} \left(2x - \frac{3}{x}\right) = -\infty$

1.10 Continuous Functions:

A function is continuous if it is continuous at each point of its domain.

Discontinuous Function:

If a function f is not continuous at a point c, we say that f is discontinuous at c and call c a point of discontinuity of f.

The continuity Test:

The function y=f(x) is continuous at x=c, if and only if all three of the following statement are true:

1. f(c) exist (c is in the domain of f).
2. $\lim_{x\to c} f(x)$ exist (f has a limit as x→c).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).

The Limit Combination Theorem for Continuous Function:

If the function f and g are continuous at x=c, then all the following combination are continuous at x=c,

$1) f+g$

$2) f-g$

$3 f.g$

$4) \frac{f}{g}$

* If a function is differentiable at a point c, then it is continuous at c as well.

Example: what value should be assigned to (a) to make the function continuous at x=3,

$$f(x)=\begin{cases} x^2-1 & x<3 \\ 2ax & x\geq 3 \end{cases}$$

Solution:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\lim_{x\to 3^-}(x^2-1) = \lim_{x\to 3^+}(2ax)$$

$$9-1=6a \Rightarrow a=\frac{8}{6}=\frac{4}{3}$$

Example: Find the value of B, if the function below is continuous.

$$f(x)=\begin{cases} x^3 & x<\frac{1}{2} \\ Bx^2 & x\geq \frac{1}{2} \end{cases}$$

Solution:

$$\lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to \frac{1}{2}^+} f(x) \Rightarrow \lim_{x\to \frac{1}{2}^-} x^3 = \lim_{x\to \frac{1}{2}^+} Bx^2 \Rightarrow \frac{1}{8} = B\frac{1}{4} \Rightarrow \therefore B=\frac{1}{2}$$

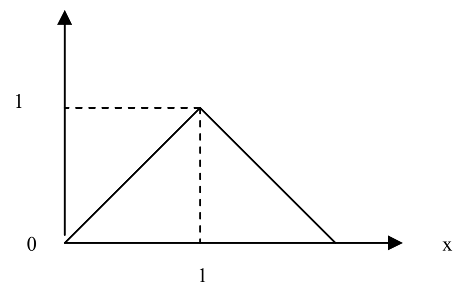
Example: Graph the function,

$$f(x)=\begin{cases} x & 0\le x\le 1 \\ 2-x & 1\le x\le 2 \end{cases}$$

Is the function, f continuous at x=1, and does it have derivative at x=1.

Solution:

x	y=2-x	y=x
0	-	0
1	-	1
1	1	-
2	0	-



The function is continuous at x=1, because

- $f(x)=x \Rightarrow f(1)=1$ and, $\lim_{x\to 1} f(x)=\lim_{x\to 1} x=1 \Rightarrow \therefore \lim_{x\to 1} x=f(1)=1$

- $f(x)=2-x \Rightarrow f(1)=1$ and, $\lim_{x\to 1} f(x)=\lim_{x\to 1}(2-x)=1 \Rightarrow \therefore \lim_{x\to 1}(2-x)=f(1)=1$

Or another way, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)$ $\Rightarrow \therefore 1=1$

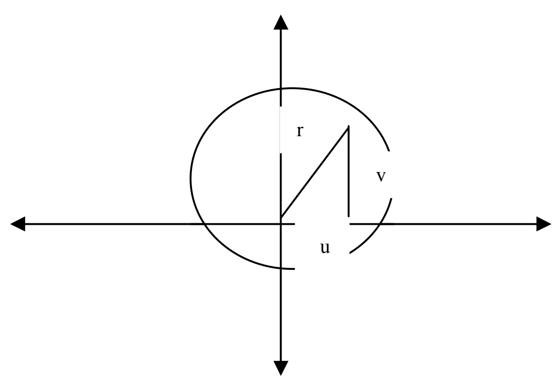
1.11 Trigonometric Functions:

Sines, Cosines, and Tangents.

$$\sin x=\frac{v}{r}$$

$$\cos x=\frac{u}{r}$$

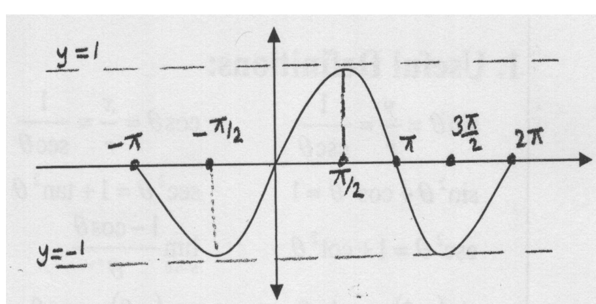
$$\tan x=\frac{\sin x}{\cos x}=\frac{v}{u}$$



The graph of function of y=sinx, y=cosx, and y=tanx, in the xy-graph are in figures below:

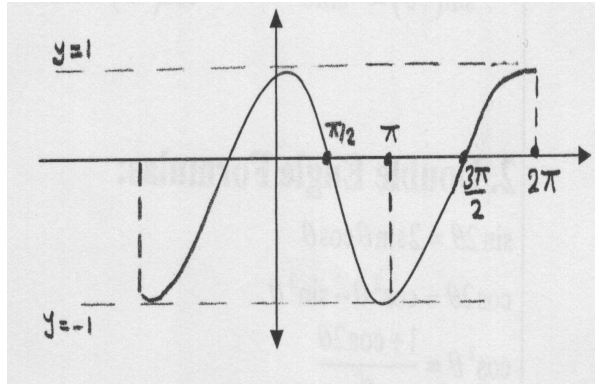
Domain -∞<x<∞ for y=sinx

Range -1≤y≤1

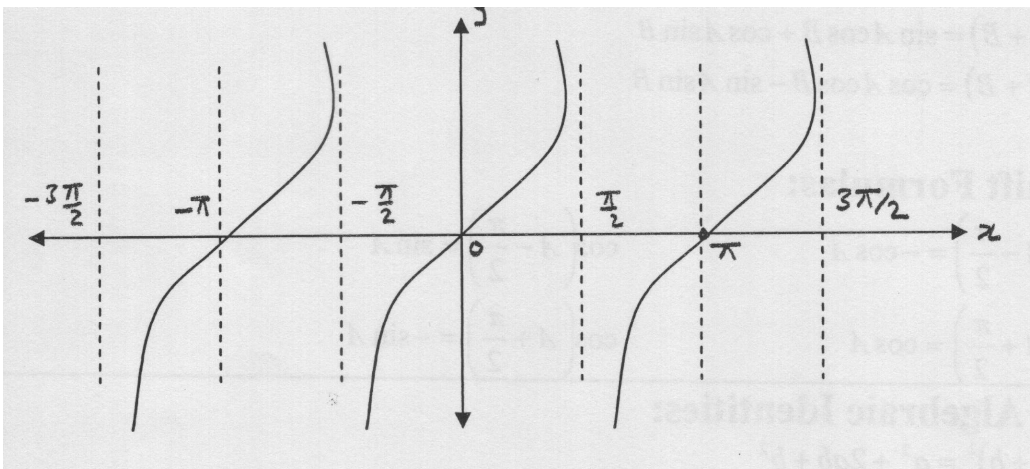


Domain $-\infty<x<\infty$ for $y=\cos x$

Range $-1\le y\le 1$



Domain for $y=\tan x$, is all the real numbers except odd integer multiples of $\frac{\pi}{2}$, and range $-\infty<y<\infty$.



Example: find the domain and range of the following functions:

1. $y=\sin^2 x$ domain all real numbers, and range $0\le y\le 1$.
2. $y=5\cos 2x$ domain, $-\infty<x<\infty$, range $-5\le y\le 5$.
3. $y=-\tan x$ domain is all the real x, except odd integer multiples of $\frac{\pi}{2}$, and range $-\infty<y<\infty$.

1.11.1 Some Useful Formulas of Trigonometric Functions:

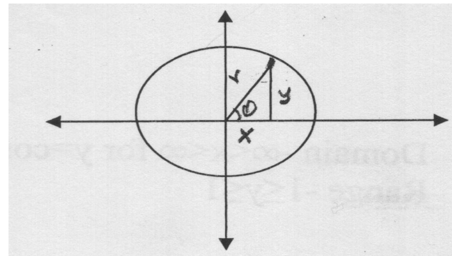
1. Useful Definitions:

$$\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{x}{r} = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{y}{x} = \frac{1}{\cot\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta$$

$$\csc^2\theta = 1 + \cot^2\theta \qquad \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$



2. Double Engle Formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

3. Sum Formulas:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

4.Shift Formulas:

$$\sin\left(A - \frac{\pi}{2}\right) = -\cos A \qquad \cos\left(A - \frac{\pi}{2}\right) = \sin A$$

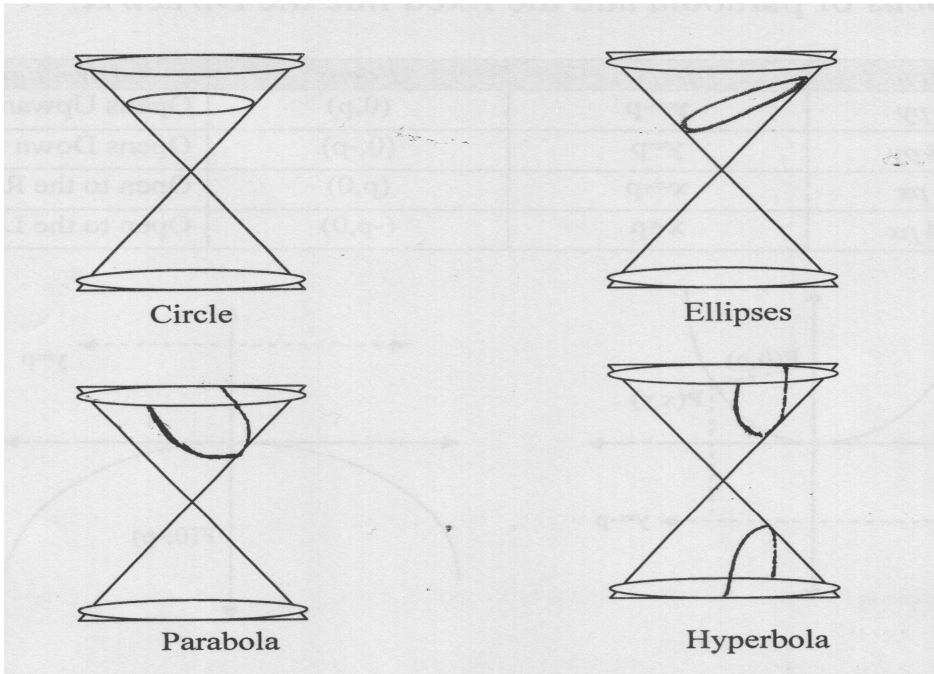
$$\sin\left(A + \frac{\pi}{2}\right) = \cos A \qquad \cos\left(A + \frac{\pi}{2}\right) = -\sin A$$

5. Algebraic Identities:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

1.12 Conical Sections:

The standard conic sections are the curves in which a plane cuts a double cone.



Circle:

A circle is the set of points in the plane whose distance from a given fixed point in the plane is a constant. The standard equation of circle of radius (a) centered at a point (h, k) is:

$$\boxed{(x-h)^2+(y-k)^2=a^2}$$

Example: Find an equation for the circle with center at the origin and with radius (a).

Solution:

$h=k=0$

$\therefore x^2+y^2=a^2$

Example: Find the circle through the origin with center at (2,-1).

Solution: center (h,k)=(2,-1)

$(x-2)^2+(y+1)^2=a^2$

Circle goes through origin, then x=y=0

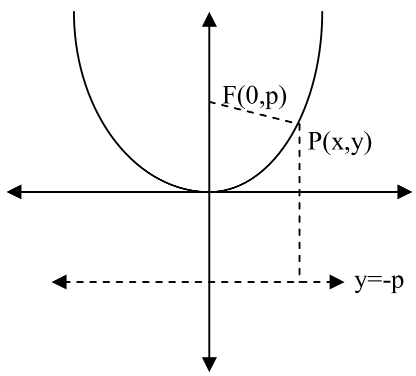
$(0-2)^2+(0+1)^2=a^2 \Rightarrow a^2=5$

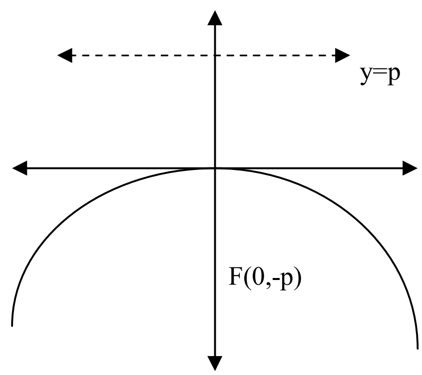
$\therefore (x-2)^2+(y+1)^2=5$

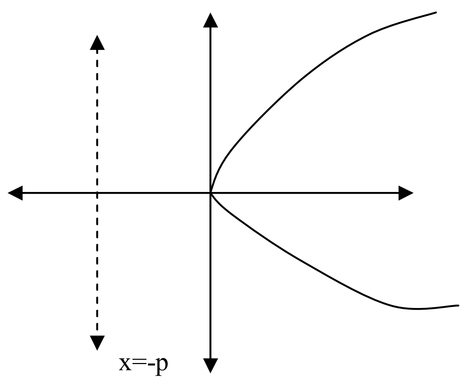
Parabolas:

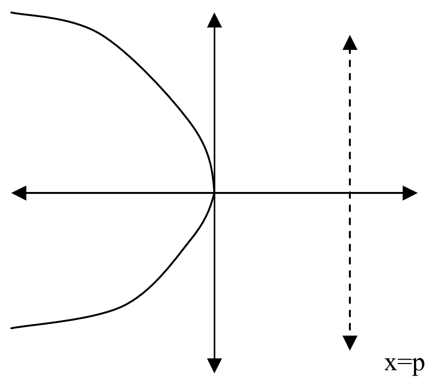
Is the set of points in the plane that are equidistant from a given fixed point and fixed line in the plane. The fixed point is called *Focus* of parabola and the fixed line the *Directrix*.

Equation	***Directrix***	***Focus***	***Direction***
$x^2 = 4py$	y=-p	(0,p)	Opens Upward
$x^2 = -4py$	y=p	(0,-p)	Opens Down ward
$y^2 = 4px$	x=-p	(p,0)	Open to the Right
$y^2 = -4px$	x=p	(-p,0)	Open to the Left









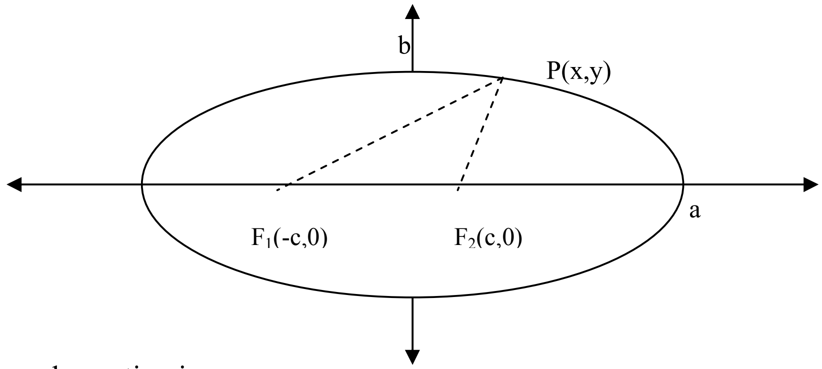
Example: Find the focus and directrix of the parabola $x^2=8y$.

Solution:

$$4p = 8 \Rightarrow p = 2$$
$$directrix, y = -p = -2 \Rightarrow \therefore\ focus, F(0,2)$$

Ellipses:

An ellipsis is the set of points in the plane whose distances from two fixed points in the plane have a constant sum.



The general equation is:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

The axes:

- $b^2 = a^2 - c^2$, b^2 is less than a^2.
- The *major axis* of ellipses of length =2a, with x-axis intercepts (±a,0).
- The *minor axis* of ellipses of length =2b, with y-intercept (0,±b).
- $\therefore\ for, \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Major axes are horizontal.
- $\therefore\ for, \frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ Major axes are vertical.

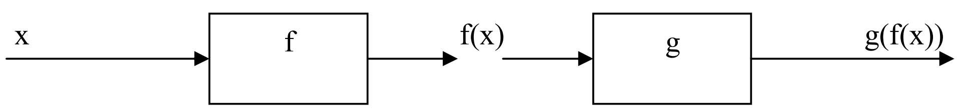
Hyperbolas:

Is the set of points in a plane whose distance from two fixed points in the plane has a constant difference.

- $\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow b^2 = c^2 - a^2 \Rightarrow focus \in, x - axies}$
- $\boxed{\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \Rightarrow b^2 = c^2 - a^2 \Rightarrow focus \in, y - axies}$

1.13 Composition of Functions:

The composition of functions produced when the outputs of function f is the input of function g.



g(f(x)) is the composite of f and g. The usual notation for this composite is f °g, which read as "g of f".

Example: If $f(x)=\sin x, and, g(x)=-\frac{x}{2}, find, g\circ f$

Solution: $g(f(x))=-\frac{f(x)}{2}=-\frac{\sin x}{2}$

Example: $f(x)=x^2, and, g(x)=x-7, find \Rightarrow g\circ f, and, f\circ g$

Solution:

$$g\circ f = g(f(x)) = f(x)-7 = x^2-7$$
$$f\circ g = f(g(x)) = (g(x))^2 = (x-7)^2$$

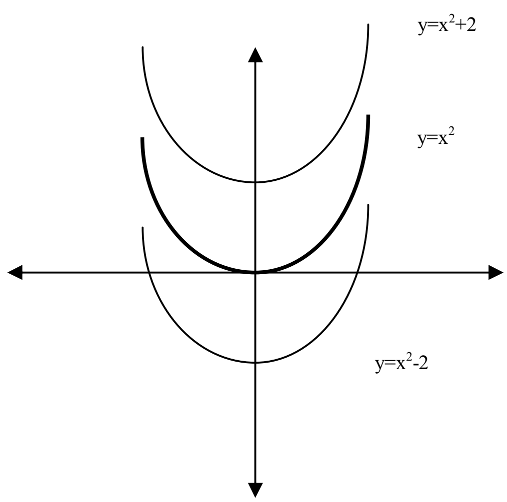
1.14 Shift Formula:

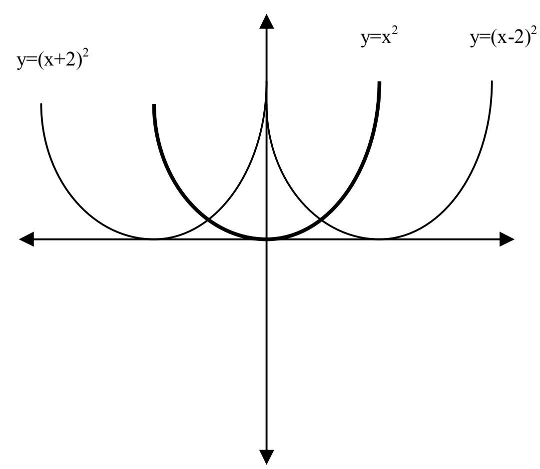
Vertical shift:

- *y=f(x)+c* shift the graph of f(x) up c units.
- *y=f(x)-c* shift the graph of f(x) down c units.

Horizontal shift:

- *y=f(x+c)* shift the graph of f(x) left c units.
- *y=f(x-c)* shift the graph of f(x) right c units.





Problems:

1. Write the equation of circle with center (5,4) that is tangent to (a) x-axis. (b) y-axis.
2. A point on the circle $x^2 + y^2 = 9$ has the property that its distance from the origin is equal to the slope of line joining it to the origin. What is the point?.
   Answer: $\left(\frac{3}{\sqrt{10}}, \frac{9}{\sqrt{10}}\right)\left(-\frac{3}{\sqrt{10}}, -\frac{9}{\sqrt{10}}\right)$
3. For what value of (k) are the two lines parallel
   $a.3x + ky = 7, and, 6x - 2y = 5 \Rightarrow Answer : k = -1$
   $b.3x + ky = 7, and, 6x - k^2 y = 5 \Rightarrow Answer : k = -2$
4. For what value of (k) is the line $kx+2y=3$, (a) Perpendicular to the line $x+y=1$. (b) Parallel to the line $x+y=1$. *Answer: (a) $k=-2$, (b) $k=2$.*
5. Find the equation of the circle with center (6,7) and tangent to the circle $(x+2)^2 + (y-1)^2 = 9$.
6. Find the points of intersection of the line $2x+2y+12=0$ with circle $x^2+y^2=65$. *Answer: (1,-8) (-7,4).*
7. Find the equation of the circle passing through the points (0,0), (4,4), and (2,6). *Answer:* $(x-1)^2+(y-3)^2=10$
8. Find the line that passes through the point (1,2) and the point of intersection of the lines $x+2y=3$ and $2x-3y=-1$. *Answer:* $x=1$
9. Find the value of k such that the line $(k-1)x+(k+1)y-7=0$ is parallel to the line $3x+5y+7=0$. *Answer:* $k=4$
10. Find the equations of all tangents to the circle $x^2+y^2-2x+8y-23=0$, (a) at (3,-10). (b) Having slope=3. *Answer:*
    $a.y = \frac{1}{3}x - 11, b.y = 3x - 27, and, y = 3x + 13$
11. Write the equation of circle passing through the points (2,3) and (-1,6), with center on the line $2x+5y+1=0$. *Answer:* $(x+3)^2 + (y-4)^2 = 29$
12. Find the value of C so that the line $y=4x+3$ is tangent to the curve $y=x^2+C$. *Answer:* $C=7$
13. What are the equations of the horizontal tangents to the curve $y=x^3-2x^2+x$. *Answer:* $y = 0, and, y = \frac{4}{27}$

14. The line x=a cuts the curve $y=\frac{1}{3}x^3+4x+1$ at point P, and the curve $y=2x^2+x-1$ at point Q. If the tangents to the curves at P & Q are parallel, what is the value of (a)? What are the equations of theses tangents?. *Answer: a=1 or a=3*
15. Find the equation of normal to the curve $y=2x^2-8x+5$ at the point where the slope is 4. *Answer:* $y=-\frac{1}{4}x-\frac{1}{4}$.
16. Find an equation of tangent to the curve $y=x^2+2x-3$ at (-2,-3). At what point is the tangent horizontal? *Answer: y=-2x-7, and (-1,-4).*
17. Find the slope of the given curves at the given points (a) $3x^2-xy+2y^2=3, at, (1,0)$. (b) $x^{\frac{3}{2}}+y^{\frac{3}{2}}=2, at(1,1)$. *Answer: a. 6, b. -1.*

Chapter Two

Derivative and Its Applications

2.1 Definitions and Rules:

Let *y=f(x)* be a function of x, then

$$\frac{dy}{dx} = f`(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If the above limit exist and finite, we call this limit the derivative of *f* at *x*, and say that f is differentiable.

The rules of derivative are:

Rule 1: The derivative of a constant is zero.

$$If, f(x) = cons\tan t = c, then, \frac{dy}{dx} = f`(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c-c}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

Rule 2: Power rule for positive integer power of x.

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Examples:

1. $\frac{d}{dx}(x) = 1 \times x^0 = 1$
2. $\frac{d}{dx}\left(x^2\right) = 2x^1 = 2x$
3. $\frac{d}{dx}\left(x^3\right) = 3x^2$
4. $\frac{d}{dx}\left(x^4\right) = 4x^3$

Rule 3: If u is any differentiable function of x and c is any constant, then:

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

Example: $\frac{d}{dx}\left(7x^5\right) = 7 \times 5x^4 = 35x^4$

Rule 4: The sum rule

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

Example: find, dy/dx if $y=x^3+7x^2-5x+4$.

$dy/dx=3x^2+14x-5$

Second derivative:

$y` = \frac{dy}{dx}$ is called first derivative function of x.

$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ the second derivative.

Example: If $y=x^3-3x^2+2$

$$y' = \frac{dy}{dx} = 3x^2 - 6x \Rightarrow y'' = \frac{d^2 y}{dx^2} = 6x - 6 \Rightarrow y''' = \frac{d^3 y}{dx^3} = 6 \Rightarrow \frac{d^4 y}{dx^4} = 0$$

2.2 Velocity and Acceleration:

Body motion $s = f(t)$

First derivative $\frac{ds}{dt}$ is body's velocity.

Second derivative $\frac{d^2 s}{dt^2}$ is body's acceleration.

Example: The position of moving body is given by the equation $s=160t-16t^2$, with s in feet and t in seconds. Find the body's velocity and acceleration at time t.

Solution:

Velocity $v = \frac{ds}{dt} = 160 - 32t$

Acceleration $\frac{dv}{dt} = \frac{d^2 s}{dt^2} = -32 \frac{ft}{\sec^2}$

Example: A heavy rock blasted vertically upward with a velocity of 160 ft/sec, reaches a height of $s=160t-16t^2$ (ft) after t seconds.

1. How high does the rock go?
2. How fast is the rock traveling when it is 256 ft above the ground on the way up? On the way down?

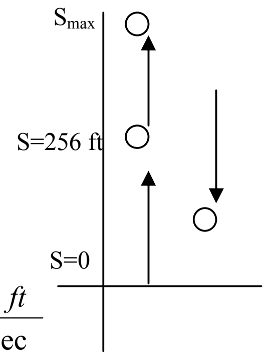
Solution:

1. To find how high the rock goes, we find the value of s when $v=0$

. $v = \frac{ds}{dt} = 160 - 32t = 0 \Rightarrow 160 - 32t = 0 \Rightarrow 160 = 32t \Rightarrow t = 5\sec$

$s = 160 \times 5 - 16 \times 5^2 \Rightarrow s_{\max} = 400\,ft$

2. To find the rock's velocity at 256 ft on the way up and again when the way down we find the tow values of t for which:



$$s(t) = 160t - 16t^2 = 256$$
$$16t^2 - 160t + 256 = 0$$
$$16(t^2 - 10t + 16) = 0 \Rightarrow (t-2)(t-8) = 0$$
$$t = 8\sec, or, t = 2\sec$$

Then, the rock is 256 ft above the ground 2 sec after explosion and again 8 sec after explosion.

then, $v(2) = 160 - 32\times 2 = 96\dfrac{ft}{\sec}, and, v(8) = 160 - 32\times 8 = -96\dfrac{ft}{\sec}$

Down velocity is –ve, because s, decreasing when t=8 sec.

Rule 5: The product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example: Find the derivative of $y = (x^2+1)(x^3+3)$.

$$\frac{dy}{dx} = (x^2+1)(3x^2) + (2x)(x^3+3)$$
$$= 3x^4 + 3x^2 + 2x^4 + 6x = 5x^4 + 3x^2 + 6x$$

Rule 6: Positive integer power of a differentiable function.

If U is a differentiable function of x, and n is positive integer, then U^n is differentiable and:-

$$\frac{d}{dx}(U^n) = nU^{n-1}\frac{dU}{dx}$$

Example: Find the derivative of $y = (x^2 - 3x + 1)^5$.

Solution:

$$y' = 5(x^2 - 3x + 1)^4(2x - 3)$$

Example: Find y` if $y = (x^2+1)^3(x-1)^2$

Solution:

$$\frac{dy}{dx} = (x^2+1)^3 \times 2(x-1)\times 1 + (x-1)^2 \times 3(x^2+1)^2 \times (2x)$$
$$= 2(x-1)(x^2+1)^3 + 6x(x-1)^2(x^2+1)^2 = 2(x-1)(x^2+1)^2(4x^2 - 3x + 1)$$

Rule 7: The Quotient Rule

If $v \neq 0$, the quotient $y = \frac{u}{v}$ of two differentiable function, then:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example: Find derivative of $y = \frac{x^2+1}{x^2-1}$.

$$\frac{dy}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Example: Find the most effective way to derive $y = \frac{1}{(x^2-1)^5}$.

$$y = (x^2-1)^{-5} \Rightarrow y' = -5(x^2-1)^{-6}(2x) = \frac{-10x}{(x^2-1)^6}.$$

This more easy than the quotient rule,

$$y' = \frac{(x^2-1)^5 \times (0) - (1)(5)(x^2-1)^4(2x)}{\left[(x^2-1)^5\right]^2} = \frac{-10x}{(x^2-1)^6}$$

Example: Do not use the quotient rule to find the derivative of,

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-2x^2-x^2+2x}{x^4} = \frac{x^3-3x^2+2x}{x^4}$$

$$= x^{-4}(x^3-3x^2+2x) = x^{-1} - 3x^{-2} + 2x^{-3}$$

$$\therefore \frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} = -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$$

Rule 8: Negative integer power of differentiable function.

$If, y = U^n \Rightarrow then, \frac{d}{dx}(U^n) = nU^{n-1}\frac{dy}{dx}$, where n is negative integer.

Proof of rule 8:

$$Let, y = u^{-m} = \frac{1}{u^m}$$

$$\therefore \frac{dy}{dx} = \frac{u^m\frac{d(1)}{dx} - (1)\frac{d(u^m)}{dx}}{(u^m)^2} = \frac{(0) - mu^{m-1}\frac{du}{dx}}{u^{2m}} = -mu^{m-1}u^{-2m}\frac{du}{dx} = -mu^{-m-1}\frac{du}{dx}$$

Example: Find $\frac{dy}{dx}, of \Rightarrow y = x^2 + \frac{1}{x^2}$.

$y = x^2 + x^{-2} \Rightarrow y' = 2x - 2x^{-3}$

Rule 9: Power rule for fractional exponents.

$$\frac{d}{dx}U^{\frac{p}{q}} = \frac{p}{q}U^{\frac{p}{q}-1}\frac{du}{dx} \qquad U \neq 0, and\ \frac{p}{q} < 1$$

Examples:

- $y = x^{\frac{1}{2}} \Rightarrow y' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
- $y = x^{\frac{1}{5}} \Rightarrow y' = \frac{1}{5}x^{\frac{-4}{5}}$
- $y = (1 - x^2)^{\frac{1}{2}} \Rightarrow y' = \frac{1}{2}(1 - x^2)^{\frac{-1}{2}}(-2x) = \frac{-x}{(1 - x^2)^{\frac{1}{2}}}$

<u>Implicit Differentiation:</u>

Example: Find $\frac{dy}{dx}, if \Rightarrow x^5 + 4xy^3 - y^5 = 2$

$$5x^4 + 4\left[x\left(3y^2\frac{dy}{dx}\right) + y^3\frac{dx}{dx}\right] - 5y^4\frac{dy}{dx} = 0$$

$$5x^4 + 12xy^2\frac{dy}{dx} + 4y^3 - 5y^4\frac{dy}{dx} = 0$$

$$(12xy^2 - 5y^4)\frac{dy}{dx} = -(5x^4 + 4y^3)$$

$$\therefore \frac{dy}{dx} = \frac{5x^4 + 4y^3}{5y^4 - 12xy^2}$$

Example: Find, y\` if

$$x^2 + y^2 = 1 \Rightarrow 2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \therefore \frac{dy}{dx} = -\frac{x}{y}$$

<u>The Chain Rule:</u>

Example: The function y=6t-10, is the composite of y=2x, and x=3t-5, how are the derivative of these three functions relate?

$$\frac{dy}{dx} = 6, \frac{dy}{dx} = 2, and\ \frac{dx}{dt} = 3$$

$$\because 6 = (2)(3)$$

$$\therefore \frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$$

Example: y=5x-2, and x=3t, find dy/dt?

$$y = 5(3t) - 2 = 15t - 2 \Rightarrow \therefore y' = 15$$

$$or, \frac{dy}{dx} = 5, and\ \frac{dx}{dt} = 3$$

Then by chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (5)(3) = 15$$

Example: Find $\frac{dy}{dt}, if \Rightarrow y = x^3 + 5x - 4, and, x = t^2 - 1, at \Rightarrow t = -1$

Solution: at $t=-1$, $x=0$

$$\frac{dy}{dx} = 3x^2 + 5$$

$$\frac{dx}{dt} = 2t$$

$$\therefore \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3x^2 + 5)(2t) = (5)(-2) = -10$$

Example: Express dy/dt in term of t if:

$y=x^3-3$, and $x=t^2-1$

$$\frac{dy}{dx} = 3x^2, and\ \frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 3x^2(2t) = 6x^2 t = 6(t^2 - 1)^2 t$$

Derivative of Trigonometric Functions:

The derivative of sine $y=\sin x$. From the definition of derivative and some useful rules:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}, note \Rightarrow$$

$$1. \sin(x+h) = \sin x \cosh + \cos x \sinh$$

$$2. \lim_{h \to 0} \frac{\sinh}{h} = 1$$

$$3. \lim_{h \to 0} \frac{\cosh - 1}{h} = 0$$

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h} = \lim_{h \to 0} \frac{\sin x(\cosh - 1) + \cos x \sinh}{h}$$

$$= \lim_{h \to 0}\left(\sin x \cdot \frac{\cosh - 1}{h}\right) + \lim_{h \to 0}\left(\cos x \cdot \frac{\sinh}{h}\right) = \sin x \cdot \lim_{h \to 0} \frac{\cosh - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sinh}{h}$$

$$= \sin x.(0) + \cos x.(1) = \cos x$$

$$\therefore y' = \cos x$$

$$\therefore \frac{d}{dx}(\sin x) = \cos x$$

If *y=sin u, andu=f(x)*, then

$$\frac{d}{dx}(\sin u) = \cos u.\frac{du}{dx}$$

Examples:

$1.y = \sin 2x \Rightarrow y' = \cos 2x.(2) = 2\cos 2x$

$2.y = \sin x^5 \Rightarrow y' = \cos x^5(5x^4) = 5x^4 \cos x^5$

$3.y = \sin^5 x \Rightarrow y' = 5\sin^4 x(\cos x) = 5\cos x \sin^4 x$

Example: Implicit differentiation

$xy + \sin y = 0$

$x\frac{dy}{dx} + y\frac{dx}{dx} + \cos y\frac{dy}{dx} = 0 \Rightarrow (x + \cos y)\frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{(x + \cos y)}$

The derivative of cosine:

$$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$$

Examples:

$1 - y = \cos 3x \Rightarrow y' = -\sin 3x(3) = -3\sin 3x$

$2 - y = \cos^2 3x \Rightarrow y' = 2\cos 3x[(-\sin 3x).(3)] = -6\cos 3x \sin 3x$

The derivative of other trigonometric functions:

Since sinx, and cosx are differentiable functions of x, then the functions,

$\tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}, \sec x = \frac{1}{\cos x}, and, \csc x = \frac{1}{\sin x}$, are also differentiable.

$\therefore 1.\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$

$2.\frac{d}{dx}(\sec u) = \sec u \tan u\frac{du}{dx}$

$3.\frac{d}{dx}(\csc u) = -\csc u \cot u\frac{du}{dx}$

$4.\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}$

Example: Find, y\` if $y=sec^2 5x$

$$y' = 2\sec 5x\left(\sec 5x \tan 5x\right)(5) = 10\sec^2 5x \tan 5x$$

$$or, y = \sec^2 5x = \frac{1}{\cos^2 5x} = (\cos 5x)^{-2}$$

$$\therefore\ y' = -2(\cos 5x)^{-3}(-\sin 5x)(5) = 10\frac{\sin 5x}{\cos^3 5x} = 10\left(\frac{\sin 5x}{\cos 5x}\right).\left(\frac{1}{\cos^2 5x}\right) = 10\sec^2 5x \tan 5x$$

Example: Find, y\` of:

$$y = \tan\sqrt{3x} = \tan(3x)^{\frac{1}{2}} \Rightarrow \therefore\ y' = \sec^2\sqrt{3x}.\left[\frac{1}{2}(3x)^{\frac{-1}{2}}(3)\right] = \frac{3\sec^2\sqrt{3x}}{2\sqrt{3x}}$$

Application of Derivative:

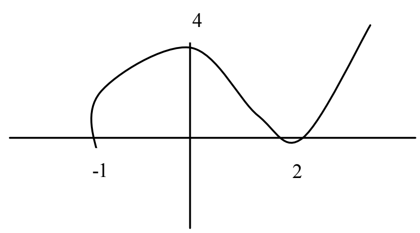
Graphing:

Example: find the intercept points with axes of the following function $y=x^3-3x^2+4$.

At x=0 y=4

At y=0 $x^3-3x^2+4=0$ *by long division*

$$\begin{array}{r|l} & x^2-4x+4 \\ \hline x+1 & x^3-3x^2+4 \\ & \underline{\mp x^3 \mp x^2} \\ & -4x^2+4 \\ & \underline{\pm 4x^2 \pm 4x} \\ & 4x+4 \\ & \underline{\mp 4x \mp 4} \\ & 0 \end{array}$$



$$(x+1)(x^2-4x+4) = 0$$

$$x = -1, x = 2$$

Then, find the sign of derivative

$y`=3x^2-6x=3x(x-2)=0$

$$\begin{array}{ccccc} ++++ & 0 & ---- & 0 & ++++ \\ & 0 & & 2 & \end{array}$$

x=0, x=2

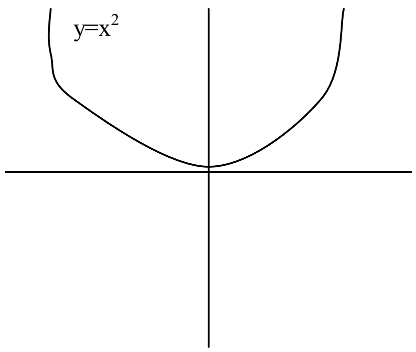
The second derivative test for concavity, if y=f(x), then

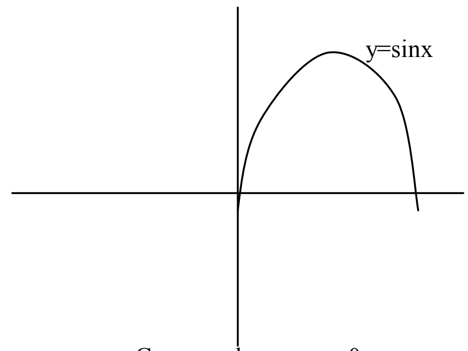
Concave down if y\`\`<0

Concave up if y\`\`>0

Example: y=x^2 y\`=2x y\`\`=2>0

Example: y=sinx y\`=cosx y\`\`=-sinx<0





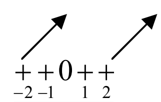
Concave down over $0<x<\pi$

Points of Inflection:

A point on the curve where the concavity changes is called a *points of inflection*. Thus, the point of inflection on twice-differentiable curve is a point where y`` is positive on one side and negative on the other. At such point y`` is zero, because derivative have the intermediate value property.

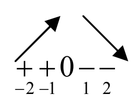
Example: $y=x^4$ $y`=4x^3$ $y``=12x^2$

$y``>0$ has no inflection points



Example:

$$y = x^{\frac{1}{3}} \Rightarrow y' = \frac{1}{3}x^{-\frac{2}{3}} \Rightarrow y'' = -\frac{2}{9}x^{-\frac{5}{3}}$$



There is a point of inflection.

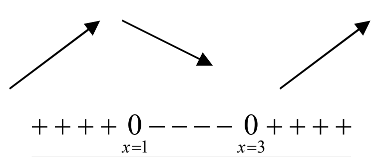
Example: Sketch the curve, $y = \frac{1}{6}\left(x^3 - 6x^2 + 9x + 6\right)$

Solution:

Calculate, y` and y`` and check the signs:

$$y' = \frac{1}{2}\left(x^2 - 4x + 3\right) = 0 \Rightarrow \therefore x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0$$

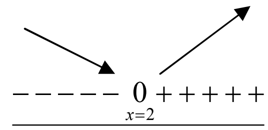
$\therefore x = 1, and, x = 3$



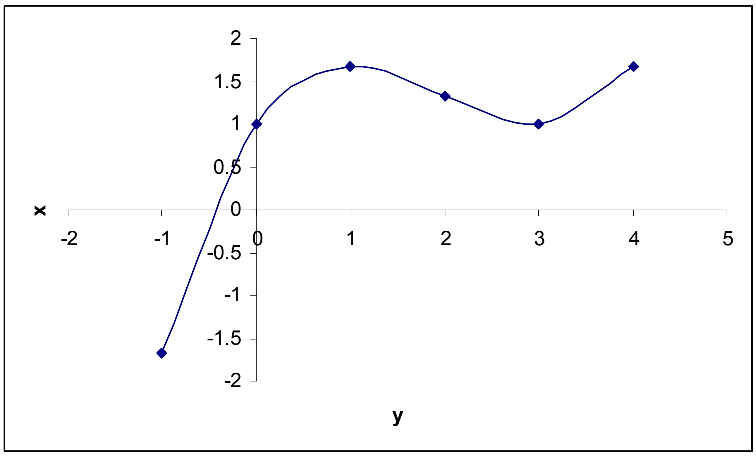
Sign of y`

This test gives information about increasing and decreasing of the curve. Then, x=1 is a local maximum point, and x=3 is local minimum point.

$$y'' = x - 2 = 0 \Rightarrow \therefore x = 2$$



Then, curve is concave down on (-∞, 2), and curve is concave up 0n (2,∞). Therefore, the point x=2 is inflection point. The graph below shows the maximum and minimum regions, and the region of inflection.

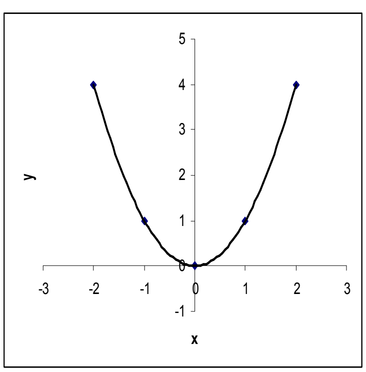


Asymptotes and Symmetry:

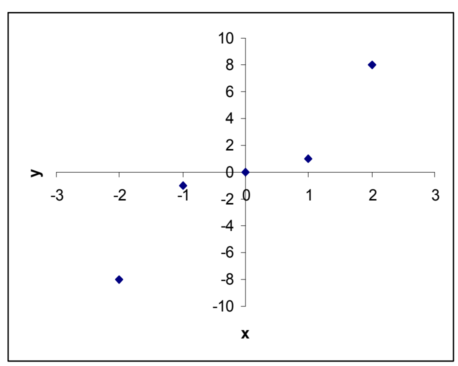
A symmetry in the graphs of even and odd functions,

y=f(x) is even if f(-x)=f(x)

y=f(x) is odd if f(-x)=-f(x) for every x in the function domain.



$y=x^2$,
$f(-x)=(-x)^2=x^2=f(x)$
f(x) is even then the graph symmetry about y-axis.



$y=x^3$
$f(-x)=(-x)^3=-x^3=-f(x)$
f(x) is odd symmetry about origin.

Example: the symmetry about y-axis, $y=(1/x^2)$.

Horizontal and Vertical Asymptotes:

- A line y=b is horizontal asymptotes of the graph of y=f(x), if

Either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

- A line x=a is a vertical asymptotes if,

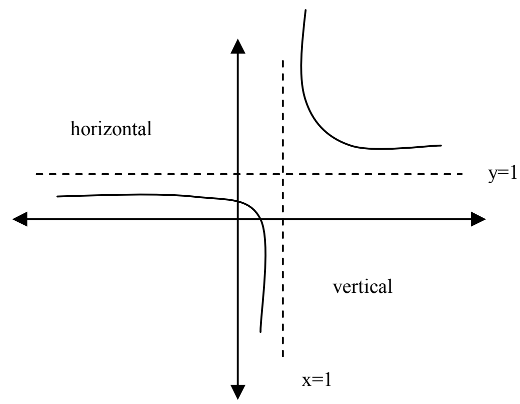
$$\lim_{x\to a^+} f(x) = \pm\infty, or, \lim_{x\to a^-} f(x) = \pm\infty$$

Example:

$$y = \frac{x}{x-1}$$

$$= \lim_{x\to\infty}\left(\frac{1}{1-\frac{1}{x}}\right) = 1$$

$$and, \lim_{x\to 1}\left(\frac{1}{1-\frac{1}{x}}\right) = \infty$$



Then, x=1 is a vertical asymptotes, and y=1 is a horizontal asymptote.

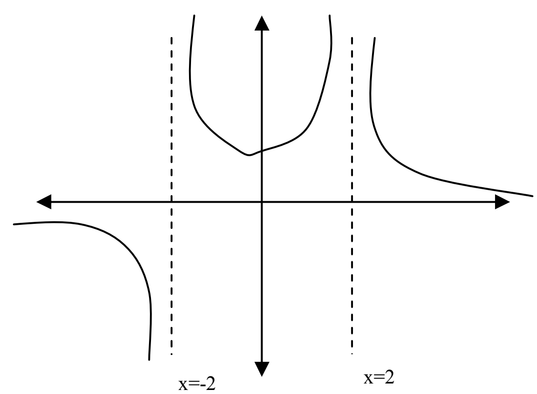
Example:

$$y = \frac{8}{4-x^2}$$

$$\lim_{x\to\infty}\left(\frac{8}{4-x^2}\right) = 0$$

$$\lim_{x\to 2^+}\left(\frac{8}{4-x^2}\right) = \infty$$

$$\lim_{x\to 2^-}\left(\frac{8}{4-x^2}\right) = -\infty$$

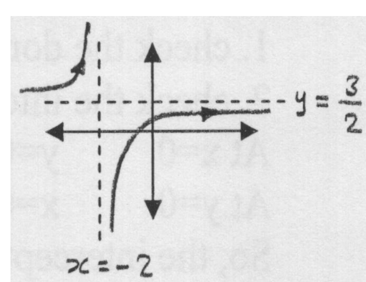


Example: $y = \frac{3x+2}{2x+4}$

We note that the domain of the function is all x/[-2], so the curve cannot intercept with vertical line x=-2, so as x approach -2, y goes to ∞, and the line x=-2 is the vertical asymptotes of the curve.

If we write the function in term of x, as $x = \frac{2-4y}{2y-3}$

So, 2y-3=0, then y=2/3 is a horizontal asymptotes.



Oblique Asymptotes:

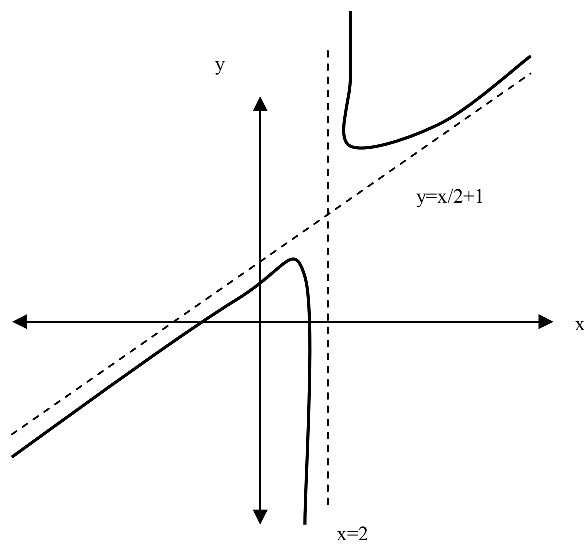
If a rational function is quotient of two polynomials that have no common factor, and the degree of numerator is one larger than the degree of the denominator, as it is in, $y=\dfrac{x^2-3}{2x-4}$, then, the graph will have an oblique asymptotes. To see why this is so, we divided (x^2-3) by (2x-4).

$$y=\frac{x^2-3}{2x-4}=\frac{x}{2}+1+\frac{1}{2x-4}$$

$$\begin{array}{r|l} & \frac{x}{2}+1 \\ \hline 2x-4 & x^2-3 \\ & \underline{\mp x^2 \pm 2x} \\ & 2x-3 \\ & \underline{\mp 2x \pm 4} \\ & 1 \end{array}$$

Vertical asymptote line x=2

Oblique asymptotes line $y=\dfrac{x}{2}+1$



Example: graph the function $y=\dfrac{x^2}{1+x^2}$.

Solution:

1. check the domain of the function $-\infty \le x \le \infty$

2. check the intercept with axises,

At x=0 y=0

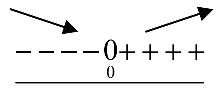
At y=0 x=0

So, the intercept point is (0,0), the origin.

3. check the sign of y` and local maximum and minimum.

$$y' = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

$$let, y' = 0 \Rightarrow \therefore 2x = 0 \Rightarrow x = 0$$

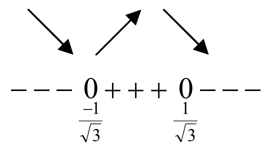


Sign of y`

Check the sign of y``:

$$y'' = \frac{-6x^2+2}{(x^2+1)^3}$$

$$let, y'' = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}} \approx \pm 0.6$$



Then, concave up between $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

And the inflection points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right) and, \left(\frac{-1}{\sqrt{3}}, \frac{1}{4}\right)$.

4. Symmetry:

$$y = f(x) = \frac{x^2}{1+x^2} \Rightarrow f(-x) = \frac{(-x)^2}{1+(-x)^2} = \frac{x^2}{1+x^2} = f(x)$$

So the function symmetry with y-axis.

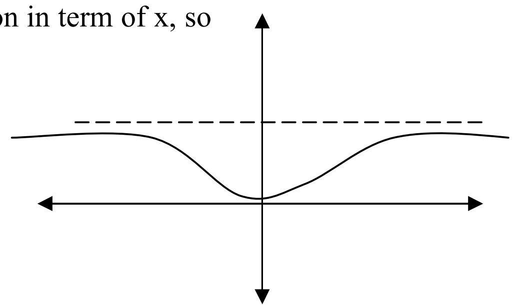
5. asymptotes:

there is no vertical asymptotes because the denominator <u>dose not equal zero.</u>

The horizontal asymptotes, by convert the function in term of x, so

$$x^2 = \frac{y}{y-1} \Rightarrow x = \sqrt{\frac{y}{y-1}}$$

So, horizontal asymptotes is y=1.



6. The graph

Example: A cylindrical container with circular base is to hold volume of (1000 cm^3), find the dimensions, so that the amount of surface area of metal required is minimum. Note that the container is open.

Solution:

$V = \pi r^2 h = 1000 .....(1)$

$A = 2\pi rh + \pi r^2 ....(2)$

$from(1)$

$\pi r^2 h = 1000 \Rightarrow h = \dfrac{1000}{\pi r^2} .....(3)$

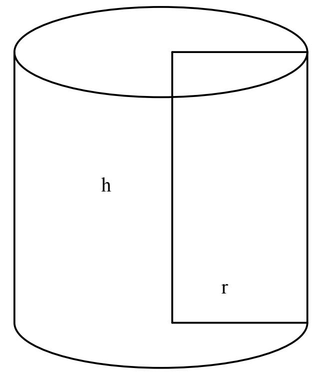
$sub.(3) in (2)$

$A = \dfrac{2000}{r} + \pi r^2 \Rightarrow \dfrac{dA}{dr} = -\dfrac{2000}{r^2} + 2\pi r = 0$

$\therefore r^3 = \dfrac{2000}{2\pi} \Rightarrow r = \sqrt[3]{\dfrac{1000}{\pi}} = \dfrac{10}{\sqrt[3]{\pi}} = 6.83cm$

$and, h = \dfrac{1000}{\pi\left(\dfrac{10}{\sqrt[3]{\pi}}\right)^2} = \dfrac{10}{\sqrt[3]{\pi}} = 6.83cm$

$\therefore h = r$



Example: Find two positive numbers whose sum is 20 and whose product is large as possible.

Solution:

Assume the first number = x

Assume the second number =20-x

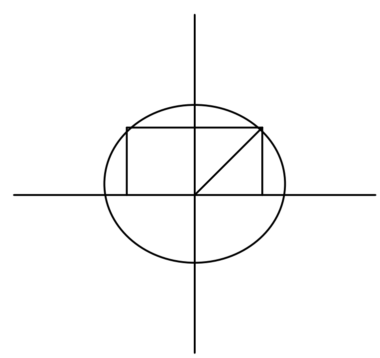
Then, the product $f(x)=x(20-x)=20x-x^2$

And, $f`(x)=20-2x=0$

Then, x=10 and 20-x=10.

Example: A rectangle is to be drawn in a semicircle of radius 2. What is the largest rectangle can have, and what are its dimensions?

Solution:



$lenght = 2x$

$height = \sqrt{4-x^2} \Rightarrow \therefore Area = A(x) = 2x\sqrt{4-x^2}$

$\dfrac{dA}{dx} = 2x.\dfrac{1}{2}(4-x^2)^{\frac{-1}{2}}.(-2x) + 2\sqrt{4-x^2} = \dfrac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} = 0$

$then, \dfrac{-2x^2 + 2(4-x^2)}{\sqrt{4-x^2}} = 0 \Rightarrow 8 - 4x^2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

$neglect - ve, then \Rightarrow A = 4 \Rightarrow \therefore length = 2x = 2\sqrt{2}, height = \sqrt{2}$

L`Hopital Rule:

To find $\lim_{x\to a}\frac{f(x)}{g(x)}$ by L`Hopital Rule**,** proceed to differentiate f and g as long as you still get the form $\frac{0}{0}, or \frac{\infty}{\infty}$ at x=a. Stop differentiation as soon as you get something else. L`Hopital Rule does not apply when either the numerator or denominator has a finite non zero limit.

Example:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\left(\frac{x}{2}\right)}{x^2} \Rightarrow \left(gives\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{1}{2}\right)(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \Rightarrow \left(still\frac{0}{0}\right)$$

$$=\lim\frac{-\left(\frac{1}{4}\right)(1+x)^{-\frac{3}{2}}}{2}=-\frac{1}{8}\Rightarrow (\therefore O.K)$$

Example:

$$\lim_{x\to 0}\frac{\sin x}{x^2}\Rightarrow\left(give\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\cos x}{2x}=\infty\Rightarrow(\therefore O.K)$$

Example:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{1+\tan x}\Rightarrow\left(\frac{0}{0}\right)\therefore\Rightarrow=\lim_{x\to\frac{\pi}{2}}\frac{\sec^2 x}{\sec^2 x}=1$$

Hint: The form ∞.0 and ∞-∞ can be sometimes be handled by change the expression to get $\frac{0}{0}, or \frac{\infty}{\infty}$ instead.

Example: $\lim_{x\to\infty}\left(x.\sin\frac{1}{x}\right)\Rightarrow(\infty.0)$ change by write x=1/t and t→0.

$$\therefore\lim_{t\to 0}\left(\frac{1}{t}.\sin t\right)=\lim_{t\to 0}\frac{\sin t}{t}=1$$

Example:

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)\Rightarrow(\infty-\infty)$$

$$=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)\Rightarrow\left(\frac{0}{0}\right)\Rightarrow=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}\Rightarrow\left(still\,\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=0\Rightarrow(\therefore OK)$$

Related Rates of Changes:

Example: How fast does the radius of spherical soap bubble changes when air is blown into it at the rate of 10 cm^3/sec?

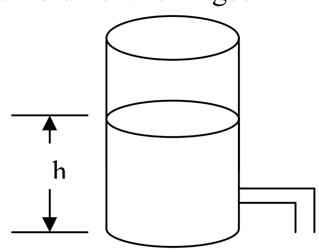
Solution:

$$\frac{dV}{dt}=10$$

$$V=\frac{4}{3}\pi r^3\Rightarrow\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}=10\Rightarrow\therefore\frac{dr}{dt}=\frac{10}{4\pi r^2}$$

Example: How fast does the water level drop when a cylindrical tank is drained at the rate of 3 lt/sec?

Solution: The radius r is constant, but the height h, and the volume V changes with time t.

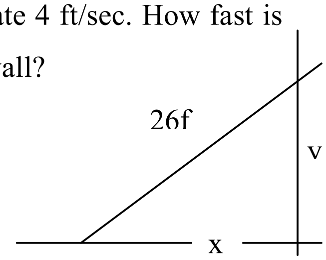


$$\frac{dV}{dt}=-3 \qquad \text{(tank drained)}$$

$$V=\pi r^2 h$$

$$\frac{dV}{dt}=\pi r^2\frac{dh}{dt}\Rightarrow\therefore\frac{dh}{dt}=-\frac{3}{\pi r^2}$$

Example: A ladder 26 ft long rest on horizontal ground and leans against vertical wall. The foot of ladder is palled away from the wall at the rate 4 ft/sec. How fast is the top sliding down the wall when the foot is 10 ft from the wall?



Solution:

$$\frac{dx}{dt}=4\frac{ft}{\sec},x=10,\frac{dy}{dt}=?$$

$$x^2+y^2=26^2\Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\therefore\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt},when,x=10\Rightarrow y=\sqrt{26^2-10^2}=24\,ft\Rightarrow\therefore\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}=-\frac{10}{24}(4)=-\frac{5}{3}\frac{ft}{\sec}$$

Chapter Three

Integration and Its Application

Integration is the opposite of differentiation process, and it divided in to two branches, *Indefinite Integrals* and *Definite Integrals*.

3.1 Indefinite Integrals:

In this type, there are no limits for integration, for example;

$$\frac{dy}{dx} = 3x^2 \Rightarrow y = \int 3x^2 dx = x^3 + C, \qquad \text{C is constant of integration.}$$

Example: Some differential equations are integrated by separation of variables, such as:

$$\frac{dy}{dx} = x^2\sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}} = x^2 dx \Rightarrow \int y^{\frac{-1}{2}} dy = \int x^2 dx$$

$$2y^{\frac{1}{2}} + C_1 = \frac{1}{3}x^3 + C_2 \quad \Rightarrow 2y^{\frac{1}{2}} = \frac{1}{3}x^3 + C$$

$$where, C = C_2 - C_1$$

Generally, the rule is:-

$$\boxed{\int U^n \frac{dU}{dx} dx = \frac{U^{n+1}}{n+1} + C}$$

Examples:

$$1.\int (5x - x^2 + 2)dx = 5\int xdx - \int x^2 dx + 2\int dx = \frac{5}{2}x^2 - \frac{1}{3}x^3 + 2x + C$$

$$2.\int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3}x^{\frac{2}{3}} + C$$

$$3.\int (x^2 + 5)^2 dx = \int (x^4 + 10x^2 + 25)dx = \frac{x^5}{5} + \frac{10x^3}{3} + 25x + C$$

$$4.\int (x^2 + 5)^2 (2x)dx = \frac{(x^2 + 5)^3}{3} + C$$

Finding the Value of Integration Constant:

Example: Find the curve whose slope at point (x,y) is ($3x^2$) and pass through the point (1,-1).

Solution:

$$Slope = 3x^2 = \frac{dy}{dx}$$
$$\therefore y = \int 3x^2 dx = x^3 + C \Rightarrow \sin ce(1,-1) \in curve \Rightarrow \therefore -1 = 1 + C \Rightarrow \therefore C = -2$$
$$\therefore y = x^3 - 2$$

Example: At time t, the velocity of moving body is given by *ds/dt=at*, a is constant, s is body's position at time t. If s=s_o, when t=0, find s?

Solution:

$$\frac{ds}{dt} = at \Rightarrow ds = \int atdt \Rightarrow s = \frac{a}{2}t^2 + C$$
$$when, t = 0, s = s_o$$
$$\therefore s_o = 0 + C \Rightarrow \therefore C = s_o$$
$$and, s = \frac{a}{2}t^2 + s_0$$

Example: If $\frac{d^2y}{dx^2} = 2 - 6x$, is a differential equation, find the function y. If we have the following initial condition, y=1 and y`=4 at x=0.

Solution:

$$\frac{dy}{dx} = 2x - 3x^2 + C_1$$
$$y = x^2 - x^3 + C_1 x + C_2$$
$$at \frac{dy}{dx} = 4, x = 0 \Rightarrow \therefore C_1 = 4$$
$$at, y = 1, x = 0 \Rightarrow \therefore C_2 = 1$$
$$\therefore y = -x^3 + x^2 + 4x + 1$$

3.2 Integrals of Trigonometric Functions:

$$1. \int \cos x dx = \sin x + C$$
$$2. \int \sin x dx = -\cos x + C$$
$$3. \int \sec^2 x dx = \tan x + C$$
$$4. \int \csc^2 x dx = -\cot x + C$$
$$5. \int \sec x \tan x dx = \sec x + C$$
$$6. \int \csc x \cot x dx = -\csc x + C$$

Examples:

$$1.\int \cos 2x dx = \frac{1}{2}\int \cos 2x(2dx) = \frac{1}{2}\sin 2x + C$$

$$2.\int \sin(7x+5)dx = \frac{1}{7}\int \sin(7x+5)(7dx) = -\frac{1}{7}\cos(7x+5) + C$$

$$3.\int \frac{\cos 2x}{\sin^3 2x}dx = \int \sin^{-3} 2x\cos 2x dx = \frac{1}{2}\int \sin^{-3} 2x\underline{\cos 2x(2dx)} = \frac{1}{2}.\frac{\sin^{(-3+1)} 2x}{(-3+1)} + C = \frac{-1}{4\sin^2 2x} + C$$

$$4.\int 16x\sin^3(2x^2+1)\cos(2x^2+1)dx \Rightarrow use, subdtitution, method$$

$$let, u = 2x^2+1, and, du = 4xdx$$

$$\therefore \int 4\sin^3 u\underline{\cos u du} = \sin^4 u + C = \sin^4(2x^2+1) + C$$

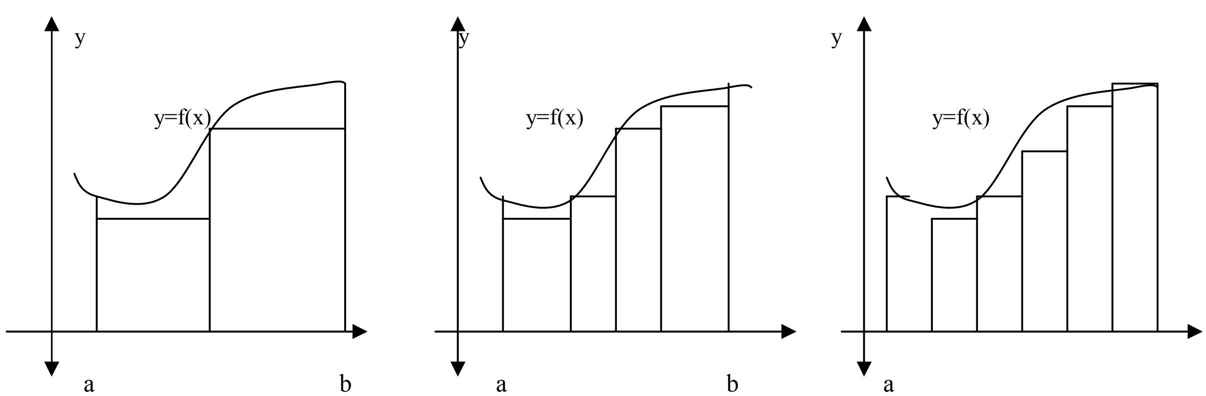
$$5.\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \int \sec^2 x dx - \int dx = \tan x - x + C$$

$$6.\int \cos^2 x dx = \int \left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}\left[\int dx + \int \cos 2x\left(\frac{2}{2}dx\right)\right] = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

3.3 Definite Integrals:

The Area Under the Curve:

To define area of the region beneath the graph of f from a to b, we approximate the region with inscribed rectangles and add the areas of these rectangles. The approximation improves as the rectangles become narrower and the number of rectangles increased.



(a) (b) (c)

Example: Estimate the area under the curve $y=x^2+1$ from a=0 to b=1 with n=4.

Solution:

$$step = \Delta x = \frac{b-a}{n} = \frac{1}{4}$$

$$Area = A_n = \sum_{n=0}^{n=4} f(x).\Delta x$$

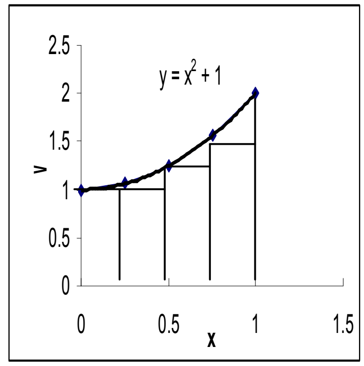
$$x_1 = \frac{1}{4} \Rightarrow f_1(x) = f_1\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + 1 = \frac{17}{16} = y_1$$

$$x_2 = \frac{2}{4} \Rightarrow y_2 = \frac{20}{16}$$

$$x_3 = \frac{3}{4} \Rightarrow y_3 = \frac{25}{16}$$

$$x_4 = \frac{4}{4} \Rightarrow y_4 = \frac{32}{16}$$

$$\therefore A_4 = \left(\frac{17}{16} + \frac{20}{16} + \frac{25}{16} + \frac{32}{16}\right)\left(\frac{1}{4}\right) = \frac{94}{64} = 1.47$$



$Exact:$

$$A = \int_0^1 \left(x^2 + 1\right)dx = \left[\frac{x^3}{3} + x\right]_0^1 = \left(\frac{1}{3} + 1\right) - (0) = \frac{4}{3} = 1.34$$

3.4 Algebraic Properties of Definite Integral:

$$1. \int_a^a f(x)dx = 0$$

$$2. \int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$3. \int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$4. \int [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$5. \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$6. \int_a^b f(x)dx \geq 0 \Rightarrow if, f(x) \geq 0, on[a,b]$$

$$7. \int_a^b f(x)dx \leq \int_a^b g(x)dx \Rightarrow if, f(x) \leq g(x)$$

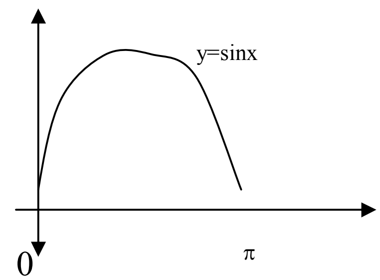
$$8. \int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Example: Calculate the area of the region enclosed by the x-axis and one arch of the curve y=sinx.

Solution:

$$\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi}$$

$$= -\cos\pi - (\cos 0) = -(-1) + 1 = 2$$



Example: Find the area between the curve $y=x^2-4$, and the x-axis from x=-2 to x=2.

Solution:

$$A = \int_{-2}^{2} (x^2 - 4)dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right) = \left|-\frac{32}{3}\right| = \frac{32}{3} unit, area$$

3.5 Rules for Approximation Definite Integrals:

1. Trapezoidal Rule:

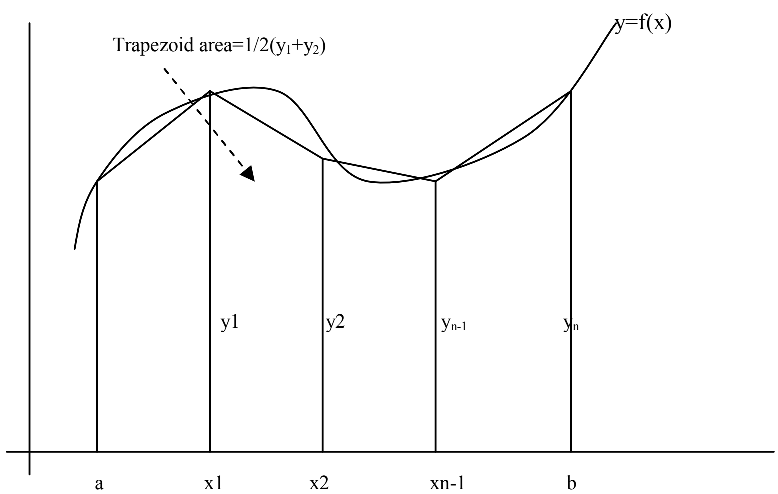
The trapezoidal rule for the value of definite integral is based on approximation the region between a curve and the x-axis with trapezoids instead of rectangles.

The sum of the areas of trapezoids is then,

$$T = \frac{1}{2}(y_o + y_1)h + \frac{1}{2}(y_1 + y_2)h + ... + \frac{1}{2}(y_{n-2} + y_{n-1})h + \frac{1}{2}(y_{n-1} + y_n)h$$

$$= h\left(\frac{1}{2}y_o + y_1 + y_2 + ... + y_{n-1} + \frac{1}{2}y_n\right) \Rightarrow where, h = \frac{b-a}{n}$$

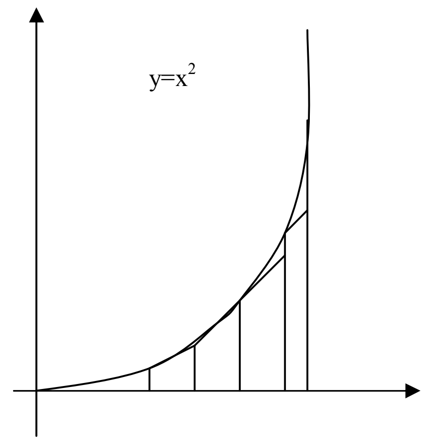
$$\boxed{\therefore T = \frac{h}{2}(y_o + 2y_1 + 2y_2 + ... + 2y_{n-1} + y_n)}$$



Example: Use trapezoidal rule with n=4 to estimate the below integral. Compare the result with the exact value.

Solution:

$$\int_1^2 x^2 dx$$



$$Exact = \int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{7}{3} = 2.3334$$

$$Trapezoidal, with, n = 4 \Rightarrow \therefore h = \frac{2-1}{4} = \frac{1}{4}$$

$$x_o = 1 \Rightarrow y_o = x^2 = (1)^2 = 1$$

$$x_1 = \frac{5}{4} \Rightarrow y_1 = \frac{25}{16}$$

$$x_2 = \frac{6}{4} \Rightarrow y_2 = \frac{36}{16}$$

$$x_3 = \frac{7}{4} \Rightarrow y_3 = \frac{49}{16}$$

$$x_4 = 2 \Rightarrow y_4 = 4$$

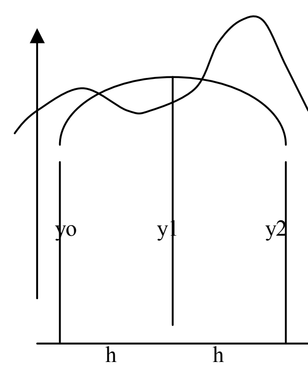
$$\therefore T = \frac{h}{2}(y_o + 2y_1 + 2y_2 + 2y_3 + y_4) = \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) = \frac{75}{32} = 2.3437$$

2. Simpson's Rule:

Simpson's rule is based on approximating curves with parabolas instead of trapezoids.

The shaded area under parabola is:

$$A_p = \frac{h}{3}(y_o + 4y_1 + y_2)$$



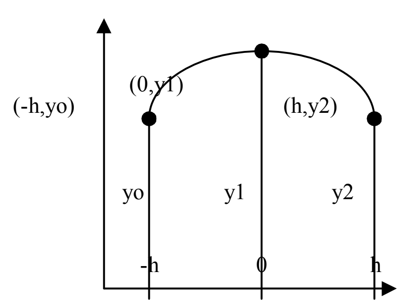
The parabola has an equation of the form:

$$y = Ax^2 + Bx + C$$

Area under it from x=-h to x=h

$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)dx = \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C)$$



Since curve pass through *(-h,y_o), (0,y_1), (h,y_2)*

$$y_o = Ah^2 - Bh + C$$

$$y_1 = C$$

$$y_2 = Ah^2 + Bh + C$$

$$\therefore C = y_1 \Rightarrow and, Ah^2 - Bh = y_o - y_1, \Rightarrow Ah^2 + Bh = y_2 - y_1$$

$$2Ah^2 = y_o + y_2 - 2y_1 \Rightarrow \therefore A_p = \frac{h}{3}\left(2Ah^2 + 6C\right) = \frac{h}{3}\left[\left(y_o + y_2 - 2y_1\right) + 6y_1\right]$$

$$\therefore Ap = \frac{h}{3}\left(y_o + 4y_1 + y_2\right)$$

$$\boxed{\begin{array}{l} Generally, S = \frac{h}{3}\left(y_o + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right) \\ n, even \Rightarrow and, h = \frac{b-a}{n} \end{array}}$$

Example: Use Simpson's rule with n=4 to approximate $\int_0^1 5x^4 dx$

Solution:

$$Exact, value = \int_0^1 5x^4 dx = \left[x^5\right]_0^1 = 1$$

$$Simpson's, Rule \Rightarrow with, n = 4 \Rightarrow \therefore h = \frac{1-0}{4} = \frac{1}{4}$$

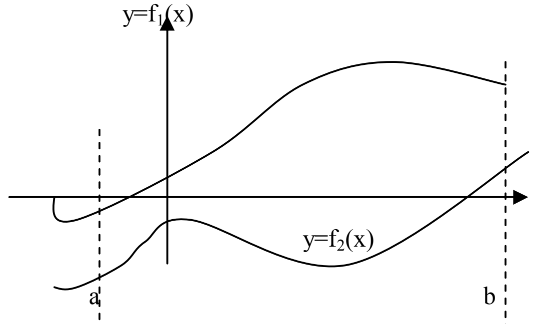
$$\therefore S = \frac{h}{3}\left(y_o + 4y_1 + 2y_2 + 4y_3 + y_4\right) = \frac{1}{12}\left[0 + 4\left(\frac{5}{256}\right) + 2\left(\frac{80}{256}\right) + 4\left(\frac{405}{256}\right) + 5\right] = 1.0026$$

3.6 Application of Definite Integral:

1. The Area Between Two Curves:

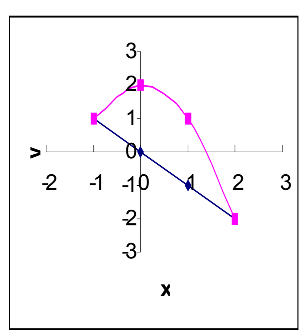
If $f_1(x) \geq f_2(x)$ through the interval $a \leq x \leq b$, then the area between the graphs of f_1 and f_2 from a to b is:

$$\boxed{Area = \int_a^b \left(f_1(x) - f_2(x)\right)dx}$$



Example: Find the area of the region bounded above by the parabola $y=2-x^2$, and below by the line $y=-x$.

Solution: Find the limits of integration,



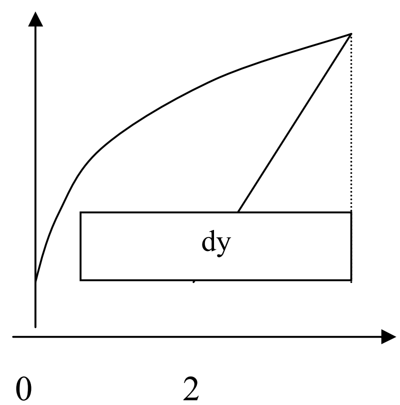
$$-x = 2 - x^2 \Rightarrow x^2 - x - 2 = 0$$
$$\therefore either, x = -1, y = 1$$
$$or, x = 2, y = -2$$
$$\therefore Area = \int_{-1}^{2}\left[\left(2 - x^2\right) - \left(-x\right)\right]dx = \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$
$$= \left[4 - \frac{8}{3} + \frac{4}{2}\right] - \left[-2 + \frac{1}{3} + \frac{1}{2}\right] = 6 - \frac{9}{3} + \frac{3}{2} = \frac{9}{2} unit, area$$

Example: Find the area of the region bounded on the right by the line y=x-2, on the left by parabola $x=y^2$, and below by the x-axis.

Solution:

Integration with respect to y:

$$y^2 = x$$
$$y + 2 = x$$
$$\therefore y^2 = y + 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0 \Rightarrow \therefore y = 2, or, y = -1, neglect$$
$$\therefore y = 2, and, y = 0$$
$$A = \int_0^2 \left(y + 2 - y^2\right)dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 = \frac{10}{3}$$



Or integration with respect to x:

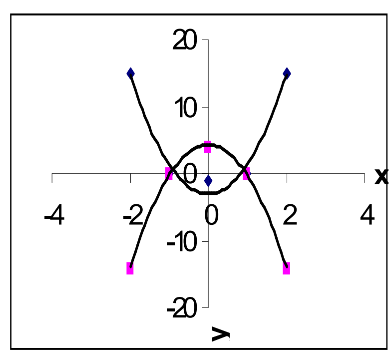
Region from 0 to 2 area under the curve.

Region from 2 to 4 area between two curves.

$$A = \int_0^2 \sqrt{x}dx + \int_2^4 \left(\sqrt{x} - x + 2\right)dx = \frac{10}{3}$$

Example: The curve $y=4-4x^2$ and $y=x^4-1$.

Solution:



$$4 - 4x^2 = x^4 - 1$$
$$x^4 + 4x^2 - 5 = 0$$
$$\left(x^2 + 5\right)\left(x^2 - 1\right) = 0$$
$$x^2 = -5, imaginary$$
$$x^2 = 1 \Rightarrow x = \pm 1$$
$$A = \int_{-1}^{1}\left[\left(x^4 - 1\right) - \left(4 - 4x^4\right)\right]dx$$
$$= \int_{-1}^{1}\left(x^4 + 4x^2 - 5\right)dx = \left[\frac{x^5}{5} + \frac{4x^3}{3} - 5x\right]_{-1}^{1} = -\frac{104}{15}$$

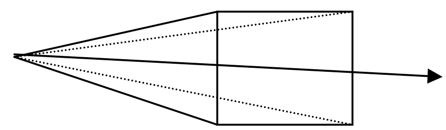
2. Volumes of Revolution:

The volume of solid of known cross-section area A(x) from x=a to x=b is:

$$\boxed{Volume = \int_a^b A(x)dx}$$

Example: A pyramid (3 m) high has a square base that is (3 m) on a side. Find the volume of pyramid.

$$Volume = \int_0^3 A(x)dx = \int_0^3 x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = 9$$



Volume of the solid of Revolution (Rotation about x-axis):

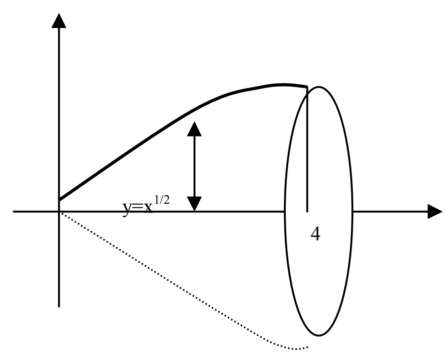
$$\boxed{Vol. = \int_a^b \pi(radius)^2\,dx = \int_a^b \pi[f(x)]^2\,dx}$$

Example: The curve $y = \sqrt{x}, where, 0 \le x \le 4$, is revolved about x-axis to generate the shape below.

Solution:

$$volume = \int_0^4 \pi(radius)^2\,dx = \int_0^4 \pi\left(\sqrt{x}\right)^2 dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi$$

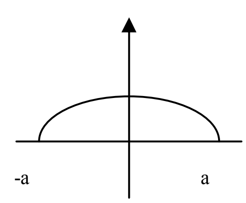


Example: The semicircle $y = \sqrt{a^2 - x^2}$, is revolved about x-axis to generate sphere. Find the volume of sphere.

Solution: Find first the points of intersection, at y=0,

$$\sqrt{a^2 - x^2} = 0 \Rightarrow x = \pm a$$

$$\therefore V = \int_{-a}^{a} \pi\left(\sqrt{a^2 - x^2}\right)^2 dx = \int_{-a}^{a} \pi\left(a^2 - x^2\right)dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4\pi a^3}{3}$$



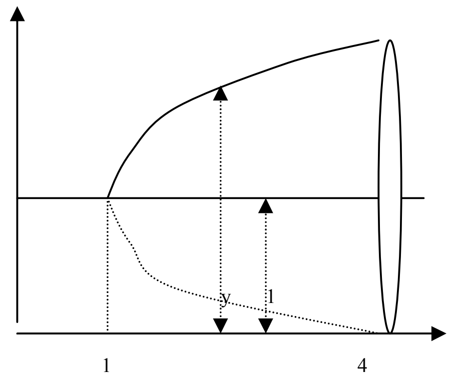
Example: Find the volume generated by revolving the region bounded by $y=x^{1/2}$, and the line y=1 and x=4 about the line y=1.

Solution:

$A(x) = \pi(radius)^2 = \pi\left(\sqrt{x} - 1\right)^2$

$Vol = \int_1^4 \pi\left(\sqrt{x} - 1\right)^2 dx = \pi\int_1^4 \left(x - 2\sqrt{x} + 1\right)dx$

$= \pi\left[\frac{x^2}{2} - \frac{4}{3}x^{\frac{3}{2}} + x\right]_1^4 = \frac{7\pi}{6}$



H.W Find the volume generated by revolving the region bounded below by the parabola $y=3x^2+1$ and above by the line y=4, about the line y=4.

Answer: $V = \frac{144}{15}\pi$.

Volume of Revolution Using Washer Method:

This method is depend on taking a rectangular strips perpendicular to axis of revolution. The formula for calculating volume by washers is:

$$\boxed{Volume = \int_a^b \pi\left[R^2(x) - r^2(x)\right]dx}$$

Where, R: outer radius, and, r: inner radius.

Example: The region bounded by the parabola $y=x^2$ and the line y=2x in the first quadrant is revolved bout the y-axis. Find the volume swept out.

Solution:

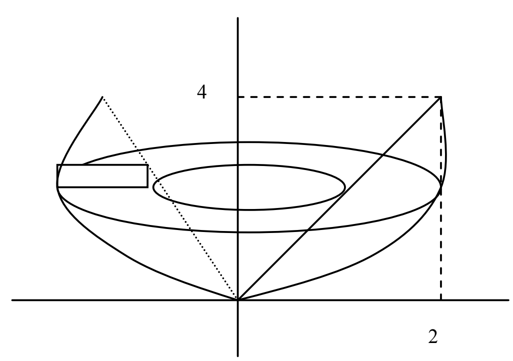
$2x = x^2 \Rightarrow \therefore x = 2, and, y = 4$

$R(y) = \sqrt{y}$

$r(y) = \frac{y}{2}$

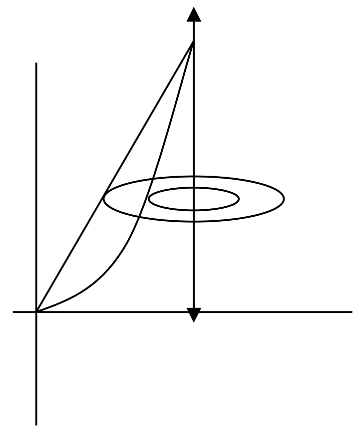
$\therefore V = \int_0^4 \pi\left(R^2 - r^2\right)dy = \int_0^4 \pi\left[\left(\sqrt{y}\right)^2 - \left(\frac{y}{2}\right)^2\right]dy$

$= \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi$



Example: The region between the parabola $y=x^2$ and the line $y=2x$, is revolved about the line $x=2$. Find the volume swept out.

Solution:



$$2x = x^2 \Rightarrow \therefore x = 2, and, y = 4$$

$$r(y) = 2 - x \;\; = \left(2 - \sqrt{y}\right)$$

$$R(y) = 2 - x = \left(2 - \frac{y}{2}\right)$$

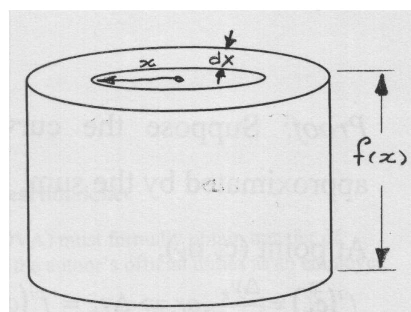
$$\therefore V = \int_0^4 \pi\left(R^2 - r^2\right)dy = \int_0^4 \pi\left[\left(2 - \frac{y}{2}\right)^2 - \left(2 - \sqrt{y}\right)^2\right]dy$$

$$= \int_0^4 \pi\left(\frac{y^2}{4} - 3y + 4\sqrt{y}\right)dy = \frac{8}{3}\pi$$

Volume of Revolution Using Cylindrical Shells Method:

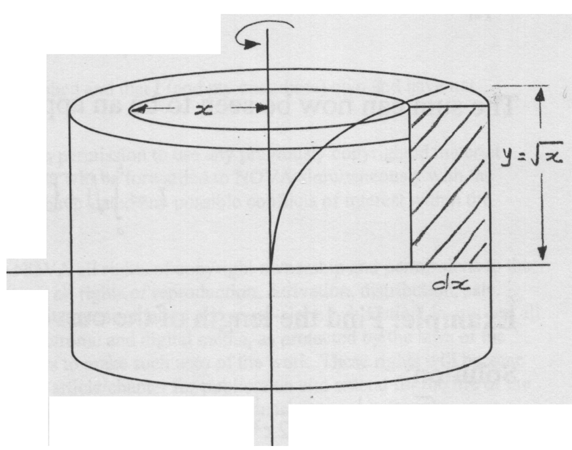
This method is depending on taking a rectangular strip parallel to axis of revolution. The formula for calculating volume by cylindrical shells is:

$$\boxed{Volume = \int_a^b 2\pi(radius)(height)dx = \int_a^b 2\pi . x . f(x)dx}$$



Example: The region bounded by $y = \sqrt{x}, y = 0, and, x = 4,$ is revolved about y-axis. Find the volume swept out.

Solution:



$$radius = x$$

$$height = f(x) = \sqrt{x}$$

$$\therefore V = \int_0^4 2\pi x\sqrt{x}dx = 2\pi\int_0^4 x^{\frac{3}{2}}dx$$

$$= 2\pi\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = \frac{128}{5}\pi$$

H.W: Use Washer method to find the volume of the region bounded by the curve $y=x^2+1$ and the line $y=-x+3$, which is revolved about the x-axis.

Answer: $V = \frac{117}{5}\pi$, unit volume

H.W: Find the volume of the solid generated when the region between the graphs of equation $y=0.5+x^2$, and $y=x$, over interval [0,2], is revolved about x-axis.

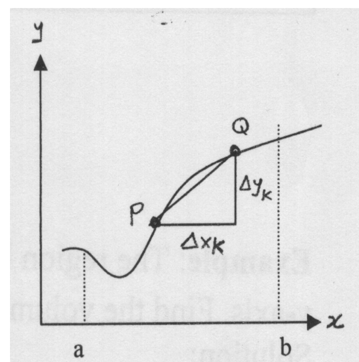
Answer: $V = \frac{69}{10}\pi$, unit volume

3. Length of Plane Curves:

Length of the curve y=f(x) from x=a to x=b is:

$$\boxed{L = \int_a^b \sqrt{1+(y')^2}\,dx}$$

Proof: Suppose the curve y=f(x) from x=a to x=b, the length is approximated by the sum, $\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.



At point (c_k,d_k),

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k}, or \Rightarrow \Delta y_k = f'(c_k)\Delta x_k$$

$$\therefore \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + \left(f'(c_k)\ \Delta x_k\right)^2} = \sum_{k=1}^{n} \sqrt{1+(f'(c_k))^2}\,(\Delta x_k)$$

The sum can now be seen to be an approximating sum for the integral:

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Example: Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1$ from x=0 to x=1.

Solution:

$$y' = \frac{4\sqrt{2}}{3}.\frac{3}{2}.x^{\frac{1}{2}} = 2\sqrt{2}x^{\frac{1}{2}}$$

$$\therefore L = \int_0^1 \sqrt{1+(y')^2}\,dx = \int_0^1 \sqrt{1+8x}\,dx = \left[\frac{2}{3}.\frac{1}{8}.(1+8x)^{\frac{3}{2}}\right]_0^1 = \frac{13}{6}\,unit, length$$

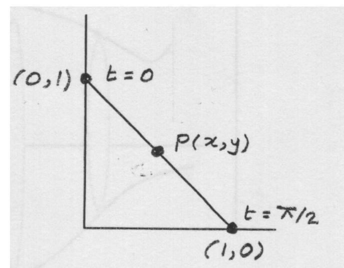
Length of Parametric Curve:

If x=x(t), y=y(t), and $a \leq t \leq b$, then

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Example: Find the distance traveled between *t=0* and $t = \frac{\pi}{2}$, by a particle p(x,y) whose position at time t is given by $x=\sin^2 t$, $y=\cos^2 t$.

Solution:



$$\frac{dx}{dt} = 2\sin t \cos t$$

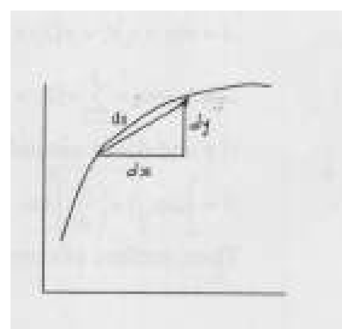
$$\frac{dy}{dt} = -2\cos t \sin t$$

$$\therefore L = \int_0^{\frac{\pi}{2}} \sqrt{(2\sin t\cos t)^2 + (-2\cos t \sin t)^2}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{8\sin^2 t \cos^2 t}\, dt$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}} 2\sin t \cos t dt = \sqrt{2}\int_0^{\frac{\pi}{2}} \sin 2t dt = \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{2}} \sin 2t(2dt)$$

$$= -\frac{\sqrt{2}}{2}[\cos 2t]_0^{\pi/2} = \sqrt{2}$$

The Short Differential Formula:



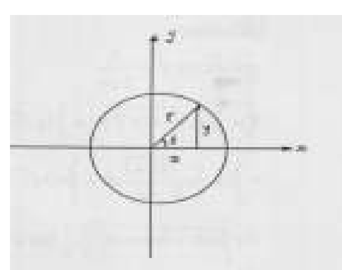
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}}\, dt = \sqrt{\frac{dx^2}{dt^2}dt^2 + \frac{dy^2}{dt^2}dt^2}$$

$$= \sqrt{dx^2 + dy^2} = ds$$

$$\therefore L = \int ds$$

Example: Show that the formula $L = \int ds$, gives the correct result for the circumference of circle of radius r.

Solution:



x=r cost

y=r sint $\quad 0 \leq t \leq 2\pi$

$$dx = -r\sin t dt \Rightarrow and, dy = r\cos t dt$$

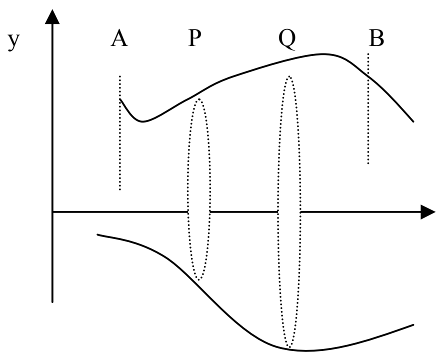
$$ds^2 = dx^2 + dy^2 = r^2\left(\sin^2 t + \cos^2 t\right)dt^2 = r^2 dt^2$$

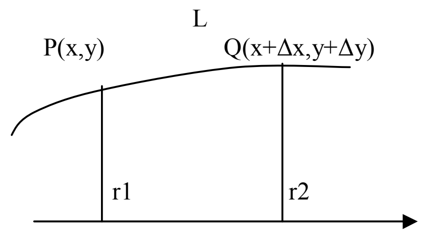
$$\therefore ds = r dt$$

$$\therefore L = \int ds = \int_0^{2\pi} r dt = [rt]_0^{2\pi} = 2\pi r$$

4. The Area of a Surface of Revolution:

Curve AB is revolved about x-axis to generate a surface.





$$\therefore r_1 = y, r_2 = y + \Delta y, L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$A = \pi(r_1 + r_2)L = \pi(2y + \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$Total, Area = \sum_{x=A}^{B} \pi(2y + \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum_{x=A}^{B} 2\pi\left(y + \frac{1}{2}\Delta y\right)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$$

If y and dy/dx are continuous function of x, the sums approach limit,

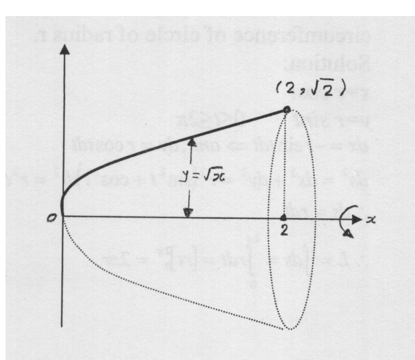
$S = \int_A^B 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$, and note we ignoring the term $\frac{1}{2}\Delta y$.

Then, surface of curve y=f(x), a≤x≤b, revolving about x-axis is:

$$\boxed{S = \int_a^b 2\pi y\sqrt{1 + (y')^2}dx}$$

Example: Find the area of the surface obtained by revolving the curve $y=x^{1/2}$, $0 \le x \le 2$.

Solution:



$$y = \sqrt{x} \Rightarrow y' = \frac{1}{2\sqrt{x}}$$

$$S = \int_0^2 2\pi y\sqrt{1+(y')^2}\,dx = \int_0^2 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}dx$$

$$= \int_0^2 2\pi\sqrt{x}\sqrt{\frac{4x+1}{4x}}dx = \int_0^2 \pi 2\sqrt{x}\,\frac{\sqrt{4x+1}}{2\sqrt{x}}dx$$

$$= \pi\int_0^2 \sqrt{4x+1}dx = \pi.\left[\frac{2}{3}.\frac{1}{4}.(4x+1)^{\frac{3}{2}}\right]_0^2 = \frac{13}{3}\pi$$

Other Form of the Integrals:

- If the axis of revolution is y-axis, the formula that replaces is:

$$\boxed{S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$$

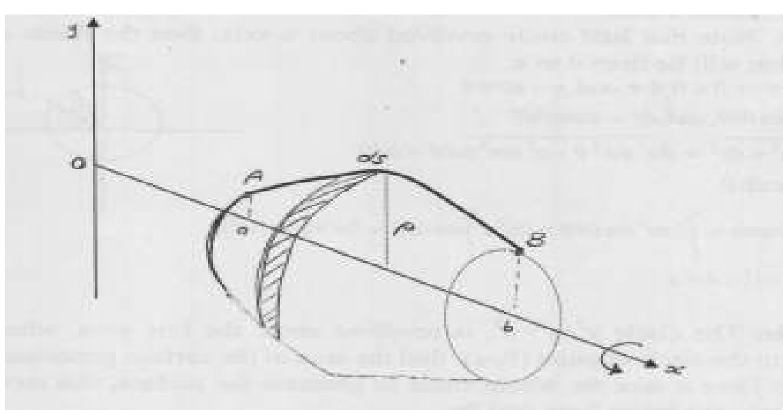
- If the curve given in parametric form, with x and y as a function of third variable t, that is from a to b, then we may compute S from the formula:

$$\boxed{S = \int_a^b 2\pi\rho\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$

ρ: is the distance from the axis of revolution to the element of arc length and is expressed as a function of t.

- The formula for surface area are all have the form:

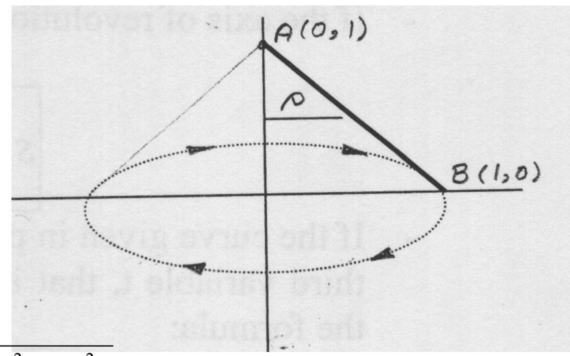
$$\boxed{S = \int 2\pi(Radius)(band, width) = \int 2\pi\rho ds}$$



H.W: Find the area of the surface that is generated by revolving the portion of the curve $y=x^2$, between x=1 to x=2 about the y-axis.

Answer: 30.85 unit area

Example: The line segment, $x=\sin^2 t$, $y=\cos^2 t$, $0 \le t \le \pi/2$, is revolved about y-axis to generate cone. Find its surface area.



Solution:

$$\rho = x = \sin^2 t$$

$$\frac{dx}{dt} = 2\sin t\cos t$$

$$\frac{dy}{dt} = -2\cos t\sin t$$

$$S = \int_a^b 2\pi\rho\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\frac{\pi}{2}} 2\pi\sin 2t\sqrt{8\sin^2 t\cos^2 t}\,dt$$

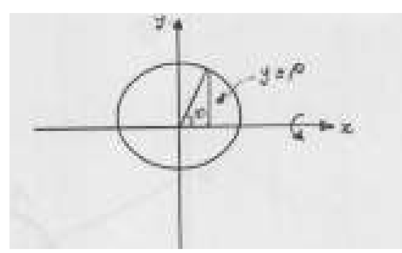
$$= 4\sqrt{2}\pi\int_0^{\frac{\pi}{2}} \sin^3 t\cos t\,dt = 4\sqrt{2}\pi\left[\frac{1}{4}\sin^4 t\right]_0^{\frac{\pi}{2}}$$

$$= \pi\sqrt{2}(1-0) = \pi\sqrt{2}$$

Example: Find the area of sphere generated by revolving the circle $x^2+y^2=a^2$, about x-axis.

Solution:

Note that half circle revolved about x-axis, then the limits of integration will be from 0 to π.



$$x = a\cos\theta \Rightarrow 0 \le \theta \le \pi, and, y = a\sin\theta$$

$$dx = -a\sin\theta d\theta, and, dy = a\cos\theta d\theta$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta = a\,d\theta$$

$$\rho = y = a\sin\theta$$

$$\therefore S = \int_0^{\pi} 2\pi\rho\,ds = \int_0^{\pi} 2\pi a^2\sin\theta\,d\theta = 2a^2\pi\int_0^{\pi}\sin\theta\,d\theta = 2a^2\pi\left[-\cos\theta\right]_0^{\pi}$$

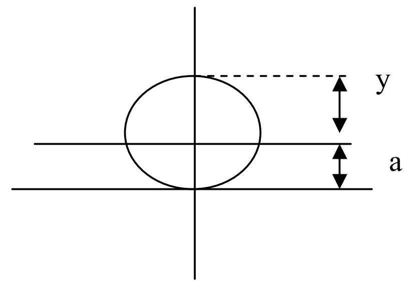
$$= 2a^2\pi[1+1] = 4a^2\pi$$

Example: The circle $x^2+y^2=a^2$, is revolved about the line y=-a, which tangent to the circle at point (0,-a). Find the area of the surface generated.

Solution:

Here it takes the whole circle to generate the surface, this mean that we take the limits from 0 to 2π.

The value of x, y, and ds, as in above example, then:



$$\rho = y + a, and, ds = \sqrt{dx^2 + dy^2} = a d\theta$$

$$\therefore S = \int_0^{2\pi} 2\pi\rho ds = \int_0^{2\pi} 2\pi(y+a) a d\theta = 2\pi \int_0^{2\pi} (a\sin\theta + a) a d\theta$$

$$= 2a^2\pi \int_0^{2\pi} (\sin\theta d\theta + d\theta) = 2a^2\pi[-\cos\theta + \theta]_0^{2\pi}$$

$$= 2a^2\pi[(-1+2\pi)-(-1+0)]$$

$$= 4a^2\pi^2$$

Chapter Four

Transcendental Functions

4.1 Inverse Function:

The inverse of the function is symbol as f^{-1}, and read "f inverse". The symbol -1 in f^{-1} is not an exponent. $f^{-1}(x)$ dose not mean *1/f(x)*.

Example: Find the inverse of $y = \frac{1}{4}x + 3$

Solution: 1st find x in term of y

$\therefore x = 4y - 12............(1)$

Then interchange x and y in equation (1):

$y = 4x - 12$

$\therefore inverse, of, y = \frac{1}{4}x + 3 \Rightarrow is, y = 4x - 12$

Example: find the inverse of function $y=8x^3$.

Solution: 1st find x in term of y, $\Rightarrow$ $x = \frac{\sqrt[3]{y}}{8^{\frac{1}{3}}} = \frac{\sqrt[3]{y}}{2}$

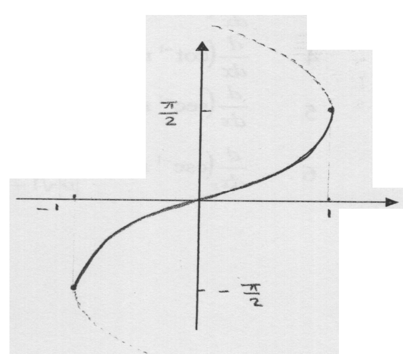
2nd interchange x and y, $\Rightarrow$ $y = \frac{\sqrt[3]{x}}{2}$ result function or the inverse function.

3rd check,

$$y = 8\left(\frac{\sqrt[3]{x}}{2}\right)^3 = 8\left(\frac{x}{8}\right) = x$$

$$y = \frac{1}{2}\sqrt[3]{8x^3} = \frac{1}{2}(2x) = x$$

4.2 The Inverse Trigonometric Functions:



$y = \sin^{-1} x$

$Dom. -1 \le x \le 1$

$Range, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Note: The (-1) in $y=sin^{-1}x$ is <u>not</u> exponent,

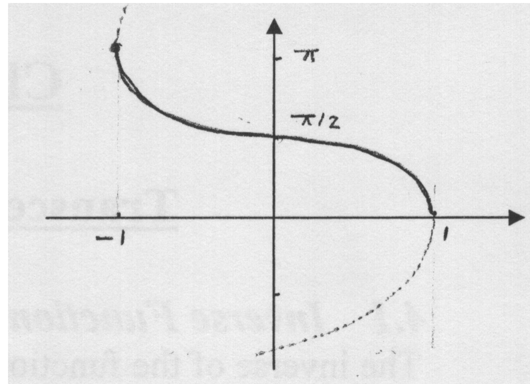
It mean inverse <u>not</u> reciprocal.

The reciprocal of sinx is:

$$(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$$

The function, *y=cosx* also has an inverse:

$$y = \cos^{-1} x$$
$$Dom. -1 \le x \le 1$$
$$Range, 0 \le y \le \pi$$



The inverse of other trigonometric functions:

1 $y = \tan^{-1} x$ — $Dom. -\infty \le x \le \infty$, $Range -\frac{\pi}{2} \angle y \angle \frac{\pi}{2}$

2 $y = \cot^{-1} x$ — $Dom. -\infty \le x \le \infty$, $Range, 0 \angle y \angle \pi$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

3 $y = \sec^{-1} x$ — $Dom. |x| \ge 1$, $Range, 0 \le y \le \pi, and, y \ne \frac{\pi}{2}$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

4 $y = \csc^{-1} x$ — $Dom. |x| \ge 1$, $Range, -\frac{\pi}{2} \le y \le \frac{\pi}{2}, and, y \ne 0$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

4.3 The Derivative of the Inverse Trigonometric Function:

1 $\frac{d}{dx}\left(\sin^{-1} u\right) = \frac{du/dx}{\sqrt{1-u^2}}$ $\quad$ *-1 < u < 1*

2 $\frac{d}{dx}\left(\cos^{-1} u\right) = -\frac{du/dx}{\sqrt{1-u^2}}$ $\quad$ *-1 < u < 1*

3 $\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{du/dx}{1+u^2}$

4 $\frac{d}{dx}\left(\cot^{-1} u\right) = -\frac{du/dx}{1+u^2}$

5 $\frac{d}{dx}\left(\sec^{-1} u\right) = \frac{du/dx}{|u|\sqrt{u^2-1}}$ $\quad$ */u /> 1*

6 $\dfrac{d}{dx}\left(\csc^{-1} u\right) = -\dfrac{du/dx}{|u|\sqrt{u^2-1}}$ $\quad /u/>1$

Example:

$y = \sin^{-1} x^2$

$y' = \dfrac{1}{\sqrt{1-\left(x^2\right)^2}} . \dfrac{d}{dx}\left(x^2\right) = \dfrac{2x}{\sqrt{1-x^4}}$

Example:

$y = \tan^{-1} \sqrt{x+1}$

$y' = \dfrac{1}{1+\left(\sqrt{x+1}\right)^2} . \dfrac{d}{dx}\left(\sqrt{x+1}\right) = \dfrac{1}{(x+2)} . \dfrac{1}{2\sqrt{x+1}} = \dfrac{1}{2\sqrt{x+1}.(x+2)}$

Example:

$y = \sec^{-1}(3x)$

$y' = \dfrac{1}{|3x|\sqrt{(3x)^2-1}} . \dfrac{d}{dx}(3x) = \dfrac{3}{|3x|\sqrt{9x^2-1}} = \dfrac{1}{|x|\sqrt{9x^2-1}}$

4.4 Integration Formula:

1 $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$ $\quad u^2 < 1$

2 $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = -\cos^{-1} u + C$

3 $\displaystyle\int \frac{du}{1+u^2} = \tan^{-1} u + C$ $\quad$ *For all u*

4 $\displaystyle\int \frac{du}{1+u^2} = -\cot^{-1} u + C$

5 $\displaystyle\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$

6 $\displaystyle\int \frac{du}{u\sqrt{u^2-1}} = -\csc^{-1} u + C$

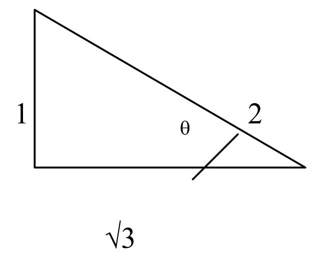
Example: $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Example: $\int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \left[\sec^{-1} x\right]_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

Note:

$$angle = \theta = \sec^{-1} \frac{2}{\sqrt{3}} \Rightarrow \therefore \sec\theta = \frac{2}{\sqrt{3}} = \frac{1}{\cos\theta}$$

$$\therefore \sec^{-1} \frac{2}{\sqrt{3}} = 30^0 = \frac{\pi}{6}$$



Example: Evaluate $\int \frac{x^2}{\sqrt{1-x^6}} dx$

Solution: Compare with

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\therefore u^2 = x^6 \Rightarrow u = x^3 \Rightarrow du = 3x^2 dx$$

$$\therefore \int \frac{x^2}{\sqrt{1-x^6}} dx = \frac{1}{3}\int \frac{3x^2 dx}{\sqrt{1-\left(x^3\right)^2}} = \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1} u + C = \frac{1}{3}\sin^{-1}\left(x^3\right) + C$$

Example: Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$

Solution:

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{9\left(1-\frac{x^2}{9}\right)}} = \int \frac{dx}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}$$

$$\therefore u = \frac{x}{3} \Rightarrow du = \frac{1}{3}dx, or, dx = 3du$$

$$\therefore \int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3du}{3\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}\left(\frac{x}{3}\right) + C$$

4.5 The Natural Logarithm:

The natural logarithm *y=ln x*

Derivative $\boxed{\frac{d}{dx}(\ln u)=\frac{1}{u}du}$

Example: find *dy/dx* if $y=\ln(3x^2+4)$

Solution: $\frac{dy}{dx}=\frac{1}{3x^2+4}(6x)=\frac{6x}{3x^2+4}$

And, integration is $\boxed{\int\frac{du}{u}=\ln|u|+C}$

Example:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\theta}{2+\sin\theta}d\theta$$

$$\therefore u=2+\sin\theta \Rightarrow du=\cos\theta d\theta$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\theta}{2+\sin\theta}d\theta=\int\frac{du}{u}=\ln|u|+C=\left[\ln(2+\sin\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\ln(2+1)-\ln(2+(-1))$$

$$=\ln 3-\ln 1=\ln 3$$

$$note, \ln 1=0$$

Example: Evaluate $\int\frac{\ln x}{x}dx$

Solution:

$$let, u=\ln x\Rightarrow du=\frac{1}{x}dx$$

$$\therefore\int\frac{\ln x}{x}dx=\int udu=\frac{u^2}{2}+C=\frac{(\ln x)^2}{2}+C$$

General formula for y=tanx and y=cotx:

$$\boxed{\begin{aligned}&\int\tan udu=-\ln|\cos u|+C=\ln|\sec u|+C\\&\int\cot udu=\ln|\sin u|+C\end{aligned}}$$

Example: Proof that $\int \tan x dx = \ln|\sec x| + C$?

Solution:

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{(-\sin x)}{\cos x} dx = -\ln|\cos x| + C$$
$$= \ln\left|(\cos x)^{-1}\right| + C = \ln|\sec x| + C$$

Example: Evaluate $\int 2x \tan(5x^2 - 1) dx$

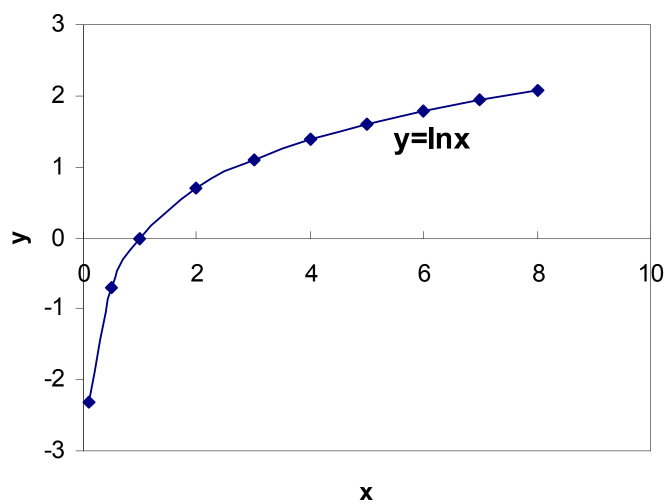
Solution:

$$\int 2x \tan(5x^2 - 1) dx = \frac{1}{5}\int \tan(5x^2 - 1)(10x dx) = \frac{1}{5}\ln\left|\sec(5x^2 - 1)\right| + C$$

4.6 Properties of Natural Logarithm:

Properties of $y=\ln x$ are:

1. Domain, *the set of positive real number, x>0.*
2. Range, *all real numbers, -∞< y <∞.*
3. $\ln(ax) = \ln a + \ln x$.
4. $\ln\left(\frac{x}{a}\right) = \ln x - \ln a$.
5. $\ln x^n = n \ln x$.
6. Note that, *ln 1=0, and (ln) of fraction is always negative.*



Examples:

$$\ln\left(\frac{1}{8}\right) = \ln 1 - \ln 8 = 0 - \ln 2^3 = -3\ln 2$$

$$\ln 4 - \ln 5 = \ln\left(\frac{4}{5}\right) = \ln(0.8) = -0.2231$$

$$\ln\sqrt[3]{25} = \ln(25)^{\frac{1}{3}} = \frac{1}{3}\ln\left(5^2\right) = \frac{2}{3}\ln 5$$

Example: find $\frac{dy}{dx}, if, y = \ln\frac{x\sqrt{x+5}}{(x-1)^3}$

Solution:

$$y = \ln x\sqrt{x+5} - \ln(x-1)^3 = \ln x + \ln\sqrt{x+5} - 3\ln(x-1) = \ln x + \frac{1}{2}\ln(x+5) - 3\ln(x-1)$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2(x+5)} - \frac{3}{(x-1)}$$

Example: find $\frac{dy}{dx}, if, y = \frac{\sqrt{\cos x}}{x^2 \sin x}$ $\quad 0 < x < \pi/2$.

Solution: to simplify differentiation process, take the (ln) for both sides;

$$\therefore \ln y = \ln\frac{\sqrt{\cos x}}{x^2 \sin x} = \ln\sqrt{\cos x} - \ln\left(x^2 \sin x\right) = \frac{1}{2}\ln\cos x - 2\ln x - \ln\sin x$$

Derivative, is :

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\cdot\frac{(-\sin x)}{\cos x} - \frac{2}{x} - \frac{\cos x}{\sin x}$$

$$\therefore \frac{dy}{dx} = y\left(-\frac{1}{2}\tan x - \frac{2}{x} - \cot x\right) = \frac{\sqrt{\cos x}}{x^2 \sin x}\left(-\frac{1}{2}\tan x - \frac{2}{x} - \cot x\right).$$

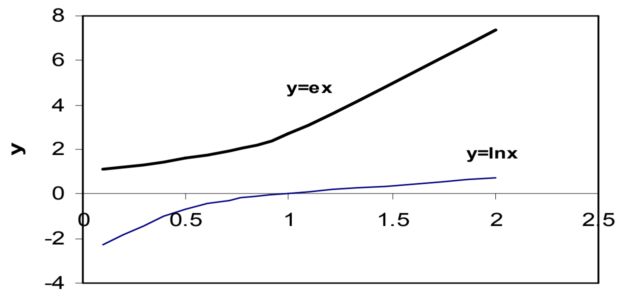
4.7 The Exponential Function e^x:

Properties of $y=e^x$:

1. The exponential function $y=e^x$ is the inverse of natural logarithm function $y=\ln x$, that is $e^x=\ln^{-1}x$.
2. Domain of $y=e^x$, $-\infty<x<\infty$.
3. Range, $y>0$
4. The derivative is, $\frac{d}{dx}\left(e^x\right) = e^x$, and if u is a function of x, then $\boxed{\frac{d}{dx}\left(e^u\right) = e^u\frac{du}{dx}}$.

5. Integration $\boxed{\int e^u du = e^u + C}$.

6. $e^{x_1}.e^{x_2} = e^{x_1+x_2}, and, e^{-x} = \frac{1}{e^x}$.



Note: Because y=e^x and y=lnx are inverse of one another, then, $e^{lnx}=x$, and $lne^x=x$.

Examples:

$\ln e^2 = 2$

$\ln\sqrt{e} = \frac{1}{2}$

$e^{\ln(x^2+1)} = x^2 + 1$

$\ln\frac{e^{2x}}{5} = \ln e^{2x} - \ln 5 = 2x - \ln 5$

$\ln e^{-1} = -1$

$e^{\ln 2} = 2$

$\ln e^{\sin x} = \sin x$

Examples: solve for y

1. lny=x^2

take (e) for both sides, $e^{\ln y} = e^{x^2} \Rightarrow \therefore\ y = e^{x^2}$

2. $e^{3y} = 2+\cos x \Rightarrow \ln e^{3y} = \ln(2+\cos x) \Rightarrow y = \frac{1}{3}\ln(2+\cos x)$.

3. solve

$$\ln(y-1) - \ln y = 3x \Rightarrow \ln\frac{y-1}{y} = 3x \Rightarrow \frac{y-1}{y} = e^{3x}$$

$$y-1 = ye^{3x} \Rightarrow y - ye^{3x} = 1 \Rightarrow y = \frac{1}{1-e^{3x}}$$

Examples: Simplify,

1.$e^{\ln 2+3\ln x} = e^{\ln 2}.e^{3\ln x} = e^{\ln 2}.e^{\ln x^3} = 2x^3$

2.$e^{-\ln x+2x} = e^{-\ln x}.e^{2x} = \frac{e^{2x}}{e^{\ln x}} = \frac{e^{2x}}{x}$

Example: Evaluate $\int_0^1 \frac{e^{\tan^{-1}x}}{1+x^2}dx$

Solution: substitute

$$u = \tan^{-1}x \Rightarrow du = \frac{dx}{1+x^2}$$
$$at, x = 0 \Rightarrow u = \tan^{-1}0 = 0$$
$$at, x = 1 \Rightarrow u = \tan^{-1}1 = \frac{\pi}{4}$$
$$\therefore \int_0^1 \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int_0^{\frac{\pi}{4}} e^u du = \left[e^u\right]_0^{\frac{\pi}{4}} = e^{\frac{\pi}{4}} - e^0 = e^{\frac{\pi}{4}} - 1$$

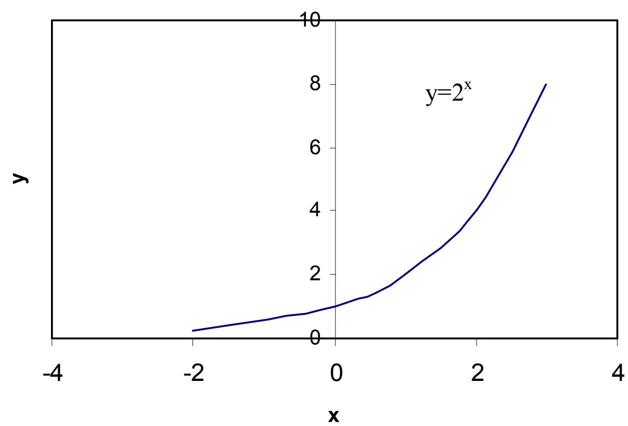
Or, substitute directly;

$$\int_0^1 \frac{e^{\tan^{-1}x}}{1+x^2}dx = \left[e^{\tan^{-1}x}\right]_0^1 = e^{\tan^{-1}1} - e^{\tan^{-1}0} = e^{\frac{\pi}{4}} - 1$$

4.8 The Function a^x and a^u:

Properties of $y=a^x$, if a is a positive real number and $a\neq 1$

1. Domain of $y=e^x$, $-\infty<x<\infty$.
2. Range, $y>0$.
3. $a^x = e^{x\ln a}$.
4. Derivative, $\frac{d}{dx}\left(a^x\right) = a^x \ln a$, if u function of x, $\boxed{\frac{d}{dx}\left(a^u\right) = a^u \ln a.\frac{du}{dx}}$.
5. Integration, $\boxed{\int a^u du = \frac{1}{\ln a}a^u + C}$.



Example: Calculate the derivative $y=3^{\sin x}$.

Solution: $y`=3^{\sin x}\ln 3 .\cos x$

Example: solve the above example by logarithm differentiation:

$$y = 3^{\sin x} \Rightarrow take, \ln$$

$$\ln y = \ln 3^{\sin x} = \sin x . \ln 3$$

$$\therefore \frac{1}{y}.\frac{dy}{dx} = \ln 3.\cos x \Rightarrow \frac{dy}{dx} = y\ln 3.\cos x = 3^{\sin x}.\ln 3.\cos x$$

4.9 Base of Logarithms:

$y = \log_a x$, can be calculated from the natural logarithm of a and x:

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Example:

$$\log_2 10 = \frac{\ln 10}{\ln 2} = \frac{2.3025}{0.6931} = 3.322$$

$$\log_4 8 = \frac{\ln 8}{\ln 4} = \frac{\ln 2^3}{\ln 2^2} = \frac{3\ln 2}{2\ln 2} = \frac{3}{2}$$

The same properties of (ln) can apply to (log):

$$\log_a uv = \log_a u + \log_a v$$

$$\log_a \frac{u}{v} = \log_a u - \log_a v$$

$$\log_a u^v = v\log_a u$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{u\ln a}.\frac{du}{dx}$$

Base 10 logarithm:

Base 10 logarithms, often called *Common Logarithm*.

$$\log_{10} x = \frac{\ln x}{\ln 10} = \frac{\ln x}{2.3026}$$

$$\therefore \ln x = 2.3026\log_{10} x$$

Example: solve the differential equation, $\frac{dy}{dx} = 2xe^{-y}$, given, at y=0, x=2.

Solution: separate the variables,

$$\frac{dy}{e^{-y}} = 2xdx \Rightarrow e^{y}dy = 2xdx \Rightarrow e^{y} = x^2 + C$$

$$use, the, condition \Rightarrow \therefore e^{0} = 2^2 + C \Rightarrow C = 1 - 4 = -3$$

$$\therefore e^{y} = x^2 - 3 \Rightarrow y = \ln\left(x^2 - 3\right)$$

Example: find, y\`, if $y=x^3e^{-2x}\cos 5x$, (Hint: use logarithmic differentiation).

Solution: take (ln) for both sides

$$\ln y = \ln\left(x^3 e^{-2x}\cos 5x\right) = \ln x^3 + \ln e^{-2x} + \ln\cos 5x = 3\ln x - 2x + \ln\cos 5x$$

$$\therefore \frac{1}{y}y' = \frac{3}{x} - 2 + \frac{(-\sin 5x)}{\cos 5x}(5) \Rightarrow y' = y\left[\frac{3}{x} - 2 - 5\tan 5x\right] = x^3e^{-2x}\cos 5x\left[\frac{3}{x} - 2 - 5\tan 5x\right]$$

Example: Evaluate,

$$\int_{\ln 3}^{\ln 5} e^{2x}dx = \frac{1}{2}\int_{\ln 3}^{\ln 5} e^{2x}(2dx) = \frac{1}{2}\left[e^{2x}\right]_{\ln 3}^{\ln 5} = \frac{1}{2}\left[e^{2\ln 5} - e^{2\ln 3}\right] = \frac{1}{2}\left[e^{\ln 5^2} - e^{\ln 3^2}\right] = \frac{1}{2}[25 - 9] = 8$$

Example: Evaluate,

$$\int_{0}^{\pi} e^{\sin x}\cos x dx = \left[e^{\sin x}\right]_0^{\pi} = e^{\sin\pi} - e^{\sin 0} = e^0 - e^0 = 0$$

Example: solve for y

$$e^{x^2}.e^{(2x+1)} = e^{y} \Rightarrow e^{\left(x^2+2x+1\right)} = e^{y} \Rightarrow y = x^2 + 2x + 1$$

Example: solve for y,

$$\ln(y-2) = \ln(\sin x) - x \Rightarrow \ln(y-2) - \ln(\sin x) = -x$$

$$\ln\frac{y-2}{\sin x} = -x \Rightarrow \frac{y-2}{\sin x} = e^{-x} \Rightarrow y - 2 = e^{-x}.\sin x \Rightarrow \underline{y = e^{-x}.\sin x + 2}$$

Chapter Five

Methods of Integration

5.1 Basic Integration Formulas

There are integrals can solved directly, like:

- Power $\int u^n du$ $\int \frac{du}{u}$
- Exponentials $\int e^u du$ $\int a^u du$
- Trigonometric Functions $\int \sin u du, \int \cos u du, \int \tan u du, \int \cot u du$

But, also, there are integrals can not solved directly. The goal of every methods of integration is to change unfamiliar integrals into integrals we recognize.

Example: Evaluate $\int \frac{dx}{1+4x^2}$

Solution: The standard form is $\int \frac{du}{1+u^2} = \tan^{-1} u + C$

$\therefore let, u = 2x, and, du = 2dx \Rightarrow dx = \frac{1}{2} du$

$$\therefore \int \frac{dx}{1+4x^2} = \int \frac{\left(\frac{1}{2}\right)du}{1+u^2} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} 2x + C$$

Example: Evaluate $\int \sqrt{1-5x}dx$

Solution: we substitute

$u = 1-5x \Rightarrow du = -5dx \Rightarrow \therefore dx = -\frac{1}{5}du$

$$\therefore \int \sqrt{1-5x}dx = \int \sqrt{u}\left(-\frac{1}{5}du\right) = -\frac{1}{5}\int \sqrt{u}du = -\frac{1}{5}.\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{2}{15}(1-5x)^{\frac{3}{2}} + C$$

Example: Evaluate $\int \frac{\sin(\ln x)}{x}dx$

Solution: Substitute $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$\therefore \int \frac{\sin(\ln x)}{x}dx = \int \sin u du = -\cos u + C = -\cos(\ln x) + C$$

Example: Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}\cos x dx$

Solution: Substitute $u = 1+\sin x \Rightarrow du = \cos x dx$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{1+\sin x}\cos x dx = \int_0^{\frac{\pi}{2}} \sqrt{u} du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}} = \frac{2}{3}\left[(1+\sin x)^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}} = \frac{2}{3}\left[(1+1)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{2}{3}\left(2\sqrt{2}-1\right)$$

Example: Evaluate $\int_0^{\frac{\pi}{4}} \cos^4 x \sin^3 x dx$

Solution:

$$\int_0^{\frac{\pi}{4}} \cos^4 x \sin^3 x dx = \int_0^{\frac{\pi}{4}} \cos^4 x(1-\cos^2 x)\sin x dx = \int_0^{\frac{\pi}{4}} (\cos^4 x - \cos^6 x)\sin x dx$$

$$\int_0^{\frac{\pi}{4}} (\cos^4 x\sin x - \cos^6 x\sin x)dx = \left[-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7}\right]_0^{\frac{\pi}{4}} = (-0+0) - \left(-\frac{1}{5}+\frac{1}{7}\right) = \frac{2}{35}$$

5.2 Integration by Parts:

The integration by parts formula is:

$$\boxed{\int udv = uv - \int vdu}$$

Example: Evaluate $\int x\cos x dx$

Solution: we use the formula $\int udv = uv - \int vdu$

$u = x \Rightarrow du = dx$

$dv = \cos x dx \Rightarrow v \sin x$

$\therefore \int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x + C$

Choosing u and dv:

For *u*: choose something becomes simpler when differentiated.

For *dv*: choose something whose integral is simple.

Example: Evaluate $\int \ln x dx$

Solution: $\int udv = uv - \int vdu$

Let $u=\ln x \Rightarrow du=(1/x)dx \Rightarrow$ (simple when differentiated)

Let $dv=dx \Rightarrow v=x \Rightarrow$ (easy to integrate)

$$\int \ln x dx = x\ln x - \int x.\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$$

Repeated Use:

Some times we have to use integration by parts more than once to obtain an answer.

Example: Evaluate $\int x^2 e^x dx$

Solution: use $\int udv = uv - \int vdu$

Let $u=x^2 \Rightarrow du=2x\,dx$

Let $dv=e^x\,dx \Rightarrow v=e^x$

$$\int x^2 e^x dx = x^2 e^x - 2\underbrace{\int_{*} xe^x dx}_{by,parts,again} \qquad \ldots (1)$$

Let U=x $\Rightarrow$ dU=dx

Let dV=e^x dx $\Rightarrow$ V=e^x

$$\int_{*} xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C^* \qquad \ldots (2)$$

Substitute equation (2) in equation (1):

$$\therefore \int x^2 e^x dx = x^2 e^x - 2\left(xe^x - e^x + C^*\right) = x^2 e^x - 2xe^x + 2e^x + C$$

Example: Evaluate $\int e^{2x}\cos 3x dx$

Solution: $\int udv = uv - \int vdu$

$$u = e^{2x} \Rightarrow du = 2e^{2x}dx$$

$$dv = \cos 3x dx \Rightarrow v = \frac{1}{3}\sin 3x$$

$$\therefore \underbrace{\int e^{2x}\cos 3x dx}_{2} = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\underbrace{\int e^{2x}\sin 3x dx}_{1}$$

*Note that the term 1 in the right side is the same as the term 2 in the left, but the difference is *sin3x* and *cos3x*, so the second choose of *u* and *dv* must gives *cos3x* inside the integration, in order to get similar terms.

Then, by parts again:

$$U = e^{2x} \Rightarrow dU = 2e^{2x}dx$$

$$dV = \sin 3x \Rightarrow V = -\frac{1}{3}\cos 3x$$

$$\therefore \underline{\int e^{2x}\cos 3x dx} = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[-\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\underbrace{\int e^{2x}\cos 3x dx}_{same, origin, \text{int} egral}\right]$$

$$\therefore \int e^{2x}\cos 3x dx + \frac{4}{9}\int e^{2x}\cos 3x dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\frac{13}{9}\int e^{2x}\cos 3x dx = \frac{3e^{2x}\sin 3x + 2e^{2x}\cos 3x}{9}$$

$$\int e^{2x}\cos 3x dx = \frac{3e^{2x}\sin 3x + 2e^{2x}\cos 3x}{13} + C$$

Tabular Integration:

Integral of the form $\int f(x)g(x)dx$, *f(x)* can be differentiated repeatedly to zero; *g(x)* can be integrated easily. Above integral can solve by parts, but it will be so cumbersome.

There is a way to organize the calculations that saves a great deal of work. It is called *Tabular Integration*.

Example: Evaluate $\int x\sec^2 x dx$ by tabular integration, and then by parts.

Solution:

<u>f(x) and its derivative</u>		<u>g(x) and its integral</u>
x	+ →	*sec²x*
1	− →	*tanx*
0		*-ln(cosx)*

$$\therefore \int x\sec^2 x dx = x\tan x + \ln(\cos x) + C$$

By parts:

$$u = x \Rightarrow du = dx$$

$$dv = \sec^2 x dx \Rightarrow v = \tan x$$

$$\int x\sec^2 x dx = x\tan x - \int \tan x dx = x\tan x - \int \frac{\sin x}{\cos x}dx = x\tan x + \ln(\cos x) + C$$

Example: Evaluate $\int x^2 e^x dx$ by tabular integration.

Solution: let $f(x)=x^2$ $g(x)=e^x$

f(x) and its derivative		g(x) and its integral
x^2	$+$	e^x
$2x$	-- -	e^x
2	$+$	e^x
0		e^x

$$\therefore \int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C$$

Example: evaluate $\int x^3 \sin x dx$

Solution: let $f(x)=x^3$ $g(x)=\sin x$

f(x) and its derivative		g(x) and its integral
x^3	$+$	$\sin x$
$3x^2$	$-$	$-\cos x$
$6x$	$+$	$-\sin x$
6	$-$	$\cos x$
0		$\sin x$

$$\therefore \int x^3 \sin x dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

H.W: Evaluate $\int x^2 \sqrt{x-1} dx$ Using tabular integration.

Answer: $\frac{2}{3} x^2 (x-1)^{\frac{3}{2}} - \frac{8}{15} x (x-1)^{\frac{5}{2}} + \frac{16}{105} (x-1)^{\frac{7}{2}} + C$

Example: Find the area of the surface generated by revolving the curve $x = e^t \sin t, y = e^t \cos t, given, 0 \le t \le \frac{\pi}{2}$, about the x-axis.

$$S = \int_a^b 2\pi\rho \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\left(\frac{dx}{dt}\right)^2 = \left(e^t \cos t + \sin t e^t\right)^2 = e^{2t} (\cos t + \sin t)^2 = e^{2t} \left(\cos^2 t + 2 \sin t \cos t + \sin^2 t\right)$$

$$= e^{2t} (1 + 2 \sin t \cos t)$$

$$\left(\frac{dy}{dt}\right)^2 = e^{2t} (1 - 2 \sin t \cos t)$$

$$\rho = y = e^t \cos t$$

$$\therefore S = 2\pi \int_0^{\frac{\pi}{2}} e^t \cos t \sqrt{e^{2t} (1 + 2 \sin t \cos t + 1 - 2 \sin t \cos t)} dt = 2\pi \int_0^{\frac{\pi}{2}} e^t \cos t \sqrt{2e^{2t}} dt = 2\sqrt{2}\pi \int_0^{\frac{\pi}{2}} e^{2t} \cos t dt$$

Integration by Parts:

$$u = e^{2t} \Rightarrow du = 2e^{2t}dt, and, dv = \cos tdt \Rightarrow v = \sin t$$

$$\int_0^{\frac{\pi}{2}} e^{2t}\cos tdt = e^{2t}\sin t - \int_* \underbrace{2e^{2t}\sin tdt}_{by,parts,again} ....(1)$$

$$\therefore U = 2e^{2t} \Rightarrow dU = 4e^{2t}dt, and, dV = \sin tdt \Rightarrow V = -\cos t$$

$$\int_* 2e^{2t}\sin tdt = -2e^{2t}\cos t + \int 4e^{2t}\cos tdt........(2)$$

$$Sub.eq.(2), in, eq.(1)$$

$$\therefore \int_0^{\frac{\pi}{2}} e^{2t}\cos tdt = e^{2t}\sin t - \left[-2e^{2t}\cos t + 4\int_0^{\frac{\pi}{2}} e^{2t}\cos tdt\right]$$

$$\int_0^{\frac{\pi}{2}} e^{2t}\cos tdt + 4\int_0^{\frac{\pi}{2}} e^{2t}\cos tdt = e^{2t}\sin t + 2e^{2t}\cos t$$

$$\int_0^{\frac{\pi}{2}} e^{2t}\cos tdt = \frac{1}{5}\left[e^{2t}(\sin t + 2\cos t)\right]_0^{\frac{\pi}{2}} = \frac{1}{5}\left[e^{\pi}(1+0) - 1(0+2)\right] = \frac{1}{5}\left(e^{\pi} - 2\right)$$

$$\therefore S = \frac{2\sqrt{2}}{5}\pi\left(e^{\pi} - 2\right)$$

5.3 Positive-Odd Power of Sine and Cosine:

Example: Evaluate $\int \sin^3 xdx$

Solution:

$$\int \sin^3 xdx = \int \sin^2 x \sin xdx = \int \left(1 - \cos^2 x\right)\sin xdx$$

$$Let, u = \cos x \Rightarrow -du = \sin xdx$$

$$\int \left(1 - u^2\right)(-du) = \int \left(u^2 - 1\right)du = \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C$$

Rule: $\boxed{\sin^{2n+1} x = \sin^{2n} \sin x = \left(\sin^2 x\right)^n \sin x}$

We then set *u=cosx*, and *–du=sinx dx*, and evaluate the integral as follow:

$$\int \sin^{2n+1} xdx = \int \left(1 - \cos^2 x\right)^n \sin xdx = -\int \left(1 - u^2\right)^n du$$

And for cosine:

$$\int \cos^{2n+1} xdx = \int \left(1 - \sin^2 x\right)^n \cos xdx = \int \left(1 - u^2\right)^n du$$

Example: Evaluate $\int \sin^7 x dx$

Solution:

$\sin^{2n+1}, \Rightarrow \therefore 2n+1=7 \Rightarrow \therefore n=3$

$$\int \sin^7 x dx = \int \left(1-\cos^2 x\right)^3 \sin x dx = \int \left(1-3\cos^2 x+3\cos^4 x-\cos^6 x\right)\sin x dx$$

$$= \int \left(\sin x - 3\sin x.\cos^2 x + 3\sin x.\cos^4 x - \sin x.\cos^6 x\right)dx$$

$$= -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

Example:

$$\int \cos^{\frac{2}{3}} x \sin^5 x dx$$

$$= \int \cos^{\frac{2}{3}} x \sin^4 x \sin x dx = \int \cos^{\frac{2}{3}} x\left(\sin^2 x\right)^2 \sin x dx = \int \cos^{\frac{2}{3}} x\left(1-\cos^2 x\right)^2 \sin x dx$$

$$= \int \cos^{\frac{2}{3}} x\left(1-2\cos^2 x+\cos^4 x\right) \ \sin x dx = \int \left(\cos^{\frac{2}{3}} x - 2\cos^{\frac{8}{3}} x + \cos^{\frac{14}{3}} x\right)\sin x dx$$

$$= -\frac{3}{5}\cos^{\frac{5}{3}} x + \frac{6}{11}\cos^{\frac{11}{3}} x - \frac{3}{17}\cos^{\frac{17}{3}} x + C$$

Products of Sine and Cosine:

The integrals of the product of sine and cosine of different angles values can be solved as:

$$\int \sin mx \sin nx dx = \int \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]dx$$

$$\int \sin mx \cos nx dx = \int \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]dx$$

$$\int \cos mx \cos nx dx = \int \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]dx$$

Example:

$$\int \sin 3x \cos 5x dx \Rightarrow \therefore m=3, n=5$$

$$= \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right]dx = \frac{1}{2}\int \left(\sin 8x - \sin 2x\right)dx = -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C$$

Example:

$$\int_0^{\pi} \cos 3x \cos 4x dx \Rightarrow m=3, n=4 = \int_0^{\pi} \frac{1}{2}\left[\cos(-x) + \cos(7x)\right]dx = \int_0^{\pi} \frac{1}{2}\left[\cos x + \cos 7x\right]dx$$

$$= \frac{1}{2}\left[\sin x + \frac{1}{7}\sin 7x\right]_0^{\pi} = \frac{1}{2}\left[\left(\sin \pi + \frac{1}{7}\sin 7\pi\right) - \sin 0\right] = \frac{1}{2}(0-0) = 0$$

Example:

$$\int \cos^5 x dx = \int \cos^4 x.\cos x dx = \int (1-\sin^2 x)^2 \cos x dx$$

$$Let, u = \sin x \Rightarrow du = \cos x dx$$

$$= \int (1-u^2) du = \int (1-2u^2+u^4) du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

The Integral of Secant and Cosecant:

$$\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1-\sin^2 x} dx = \int \frac{\cos x}{(1-\sin x)(1+\sin x)} dx$$

$$= \frac{1}{2}\int \left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x} \right) dx = \frac{1}{2}\left[-\ln|1-\sin x| + \ln|1+\sin x|\right] + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}.\frac{1+\sin x}{1+\sin x}\right| + C = \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C = \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right| + C = \frac{1}{2}\ln\left|\left(\frac{1+\sin x}{\cos x}\right)^2\right| + C$$

$$= \ln\left|\left(\frac{1+\sin x}{\cos x}\right)\right| + C = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C$$

$$\therefore \int \sec x dx = \ln|\sec x + \tan x| + C$$

A similar integration for csc x:

$$\int \csc x dx = -\ln|\csc x + \cot x| + C$$ **H.W**

Example: $\int \sec^3 x dx$

Solution: by parts

$$u = \sec x \Rightarrow du = \sec x \tan x dx$$

$$dv = \sec^2 x dx \Rightarrow v = \tan x$$

$$\because \int u dv = uv - \int v du$$

$$\therefore \int \sec^3 x dx = \sec x \tan x - \int \tan^2 x \sec x dx = \sec x \tan x - \int (\sec^2 x - 1)\sec x dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$\therefore 2\int \sec^3 x dx = \sec x \tan x + \int \sec x dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\therefore \int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Example: $\int \tan^4 x dx$

Solution:

$$\int \tan^4 x dx = \int \tan^2 x \tan^2 x dx = \int \tan^2 x (\sec^2 x - 1) dx = \int \tan^2 x \sec^2 x dx - \int \tan^2 x dx$$

$$= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1)\ x dx = \int \tan^2 x \sec^2 x dx - \int \sec^2 x dx + \int dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

5.4 The Even Power of Sine and Cosine:

We see how the identities like $\sin^2\theta = \frac{1-\cos 2\theta}{2}, and, \cos^2\theta = \frac{1+\cos 2\theta}{2}$, and other relation can help us to evaluate integrals.

Example:

$$\int \cos^4 x dx = \int (\cos^2 x)^2 dx = \int \frac{1}{4}(1+\cos 2x)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(\cos 4x + 1)\right) dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Example: solve,

$$\int \sin^2 x \cos^4 x dx = \int (1-\cos^2 x)\cos^4 x dx = \underbrace{\int \cos^4 x dx}_{last, example} - \int \cos^6 x dx$$

$$\therefore \int \cos^6 x dx = \int (\cos^2 x)^3 dx = \frac{1}{8}\int (1+\cos 2x)^3 dx = \frac{1}{8}\int \left(1 + 3\cos 2x + 3\underbrace{\cos^2 2x}_{\left(\frac{1+\cos 4x}{2}\right)} + \underbrace{\cos^3 2x}_{\substack{\cos^2 2x \cos 2x \\ (1-\sin^2 2x)\cos 2x}}\right) dx$$

we now know how to handle each term of integral. The result is:

$$\int \cos^6 x dx = \frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C,$$

Combine with the result of last example:

$$\int \sin^2 x \cos^4 x dx = \int \cos^4 x dx - \int \cos^6 x dx = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

Example: Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$

Solution: to eliminate the square root we use the identity;

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \Rightarrow 1 + \cos 2\theta = 2\cos^2\theta$$

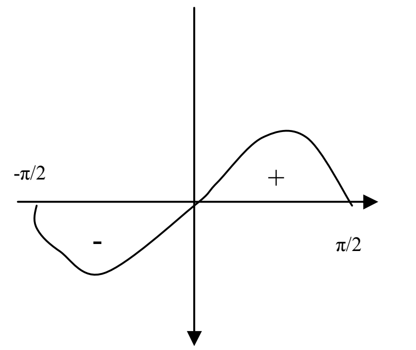
$$by, comparsion: 1 + \cos 4x = 2\cos^2 2x$$

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \sqrt{\cos^2 2x}\,dx = \sqrt{2}\int_0^{\frac{\pi}{2}} |\cos 2x|\,dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x\,dx$$

$$\cos 2x \geq 0, on \left[0, \frac{\pi}{4}\right]$$

$$\sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x\,dx = \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}(1-0) = \frac{\sqrt{2}}{2}$$

Example: Evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 t}\,dt$



Solution: we use $sin^2 t=1-cos^2 t$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 t}\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin^2 t}\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t|\,dt$$

$$|\sin t| = \begin{cases} -\sin t \Rightarrow -\frac{\pi}{2} \leq t \leq 0 \\ \sin t \Rightarrow 0 \leq t \leq \frac{\pi}{2} \end{cases}$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t|\,dt = \int_{-\frac{\pi}{2}}^{0} -\sin t\,dt + \int_0^{\frac{\pi}{2}} \sin t\,dt = [\cos t]_{-\frac{\pi}{2}}^{0} - [\cos t]_0^{\frac{\pi}{2}} = (1-0)-(0-1) = 2$$

5.5 Trigonometric Substitution: that Replace a^2-u^2, a^2+u^2, and u^2-a^2, by Single Squared Terms:

Trigonometric substitutions enable us to place binomials a^2-u^2, a^2+u^2, *and* u^2-a^2 by single squared terms. The substitutions most commonly used are:

- $u = a\sin\theta$
  $\therefore a^2-u^2 = a^2-a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$
- $u = a\tan\theta$
  $\therefore a^2+u^2 = a^2+a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$
- $u = a\sec\theta$
  $\therefore u^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta-1) = a^2\tan^2\theta$

Example: Evaluate $\int \frac{du}{a^2+u^2}$

Solution:

$$let, u = a\tan\theta \Rightarrow du = a\sec^2\theta d\theta, and, \theta = \tan^{-1}\frac{u}{a}$$

$$\therefore a^2+u^2 = a^2 + a^2\tan^2\theta = a^2\left(1+\tan^2\theta\right) = a^2\sec^2\theta$$

$$\therefore \int \frac{du}{a^2+u^2} = \int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{\theta}{a} + C = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

Example: Evaluate $\int \frac{du}{\sqrt{a^2-u^2}}$ $a>0$

Solution:

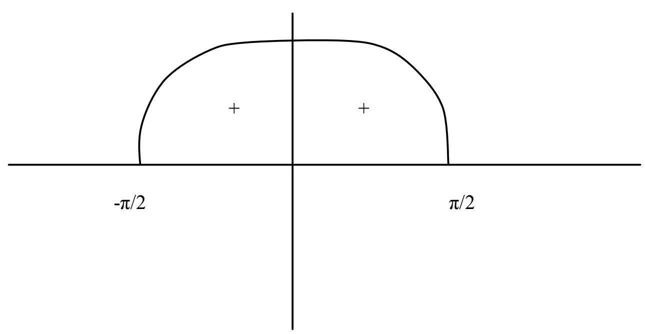
$$u = a\sin\theta \Rightarrow du = a\cos\theta d\theta, and, \theta = \sin^{-1}\frac{u}{a}$$

$$for, \sin^{-1}, the, domain, is, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\therefore a^2-u^2 = a^2 - a^2\sin^2\theta = a^2\left(1-\sin^2\theta\right) = a^2\cos^2\theta$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{a\cos\theta d\theta}{\left|a\cos\theta\right|} = \int \frac{a\cos\theta d\theta}{a\cos\theta} \Rightarrow a\cos\theta \ge 0, for, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$= \int d\theta = \theta + C = \sin^{-1}\frac{u}{a} + C$$



Example: $\int \frac{du}{\sqrt{a^2+u^2}}$ $a>0$

Solution:

$u = a\tan\theta \Rightarrow du = a\sec^2\theta d\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

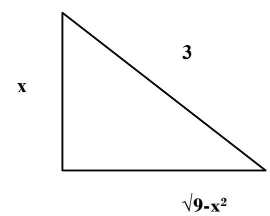
$\therefore a^2+u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$

$\int \frac{du}{\sqrt{a^2+u^2}} = \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2\sec^2\theta d\theta}} = \int \frac{a\sec^2\theta d\theta}{|a\sec\theta d\theta|} \Rightarrow \sec\theta \rangle 0, for, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$= \int a\sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{a^2+u^2}}{a} + \frac{u}{a}\right| + C$

$= \ln\frac{1}{a}\left|\sqrt{a^2+u^2}+u\right| + C = \ln\left|\sqrt{a^2+u^2}+u\right| + C'$

$where, C' = C - \ln a$



$u = a.\tan\theta \Rightarrow \tan\theta = \frac{u}{a}$

$\sec\theta = \frac{\sqrt{a^2+u^2}}{a}$

Example: Evaluate $\int \frac{x^2dx}{\sqrt{9-x^2}}$

Solution:

$like, a^2 - u^2 \Rightarrow 9 - x^2 \Rightarrow (3)^2 - x^2$

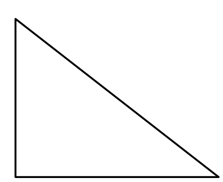
$\therefore x = 3\sin\theta \Rightarrow dx = 3\cos\theta d\theta, and, \theta = \sin^{-1}\frac{x}{3} \Rightarrow \therefore -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta$

$\int \frac{x^2dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta.3\cos\theta d\theta}{|3\cos\theta|} = 9\int \sin^2\theta d\theta = 9\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta$

$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C = \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$

$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}.\frac{\sqrt{9-x^2}}{3}\right) + C = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}.\sqrt{9-x^2} + C$



$\sin\theta = \frac{x}{3}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

5.6 Integrals Involving ax^2+bx+c:

By using the algebraic process called *Completing the Square*, we can convert any quadratic ax^2+bx+c, $a\neq 0$, to the form $a(U^2 \pm A^2)$, and then we can use one of trigonometric substitutions.

Example: completing the square of:

$$4x^2+4x+2=4(x^2+x)+2=4\left(x^2+x+\frac{1}{4}-\frac{1}{4}\right)+2=4\left(x^2+x+\frac{1}{4}\right)+2-1$$
$$=4\left(x+\frac{1}{2}\right)^2+1=4\left[\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right]=4\left(U^2+A^2\right)$$

Example: completing the square:

$$2x^2+4-6x=2x^2-6x+4=2(x^2-3x)+4=2\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)+4$$
$$=2\left(x^2-3x+\frac{9}{4}\right)+4-\frac{9}{2}=2\left(x-\frac{3}{2}\right)^2-\frac{1}{2}=2\left[\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]=2\left(U^2-A^2\right)$$

Example: Evaluate $\displaystyle\int\frac{dx}{\sqrt{2x-x^2}}$

Solution: first, completing the square of:

$$2x-x^2=-(x^2-2x)=-(x^2-2x+1-1)=-(x^2-2x+1)+1$$
$$=-(x-1)^2+1=-\left[(x-1)^2-1\right]=-\left[u^2-1\right]=1-u^2$$
$$\therefore\int\frac{dx}{\sqrt{2x-x^2}}=\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+C=\sin^{-1}(x-1)+C$$

Example: Evaluate $\displaystyle\int\frac{dx}{4x^2+4x+2}$

Solution: completing the square of:

$$4x^2+4x+2=4(x^2+x)+2=4\left(x^2+x+\frac{1}{4}\right)+1=4\left[\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right]=4\left[u^2+\left(\frac{1}{2}\right)^2\right]$$

$$\therefore \int \frac{dx}{4x^2+4x+2} = \frac{1}{4}\int \frac{du}{u^2+\left(\frac{1}{2}\right)^2}$$

$$use, trigonometric, substitution \Rightarrow u = a\tan\theta \Rightarrow du = a\sec^2\theta d\theta \Rightarrow a = \frac{1}{2}$$

$$\therefore \frac{1}{4}\int \frac{du}{u^2+\left(\frac{1}{2}\right)^2} = \frac{1}{4}\int \frac{du}{u^2+a^2} = \frac{1}{4}\int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta} = \frac{1}{4}\int \frac{d\theta}{a} = \frac{1}{4}\left[\frac{\theta}{a}+C\right]$$

$$= \frac{1}{4}.\frac{1}{a}.\tan^{-1}\left(\frac{u}{a}\right)+C = \frac{1}{2}\tan^{-1}(2x+1)+C$$

Generally: The rule for completing the square:-

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x\right)+c = a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a}$$

$$= a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right] = a\left[u^2 \pm A^2\right]$$

$$where, u = x+\frac{b}{2a}, and, \pm A^2 = \frac{4ac-b^2}{4a^2}$$

H.W

1. $\displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{3-2x-x^2}} = \frac{\pi}{6}$
2. $\displaystyle\int \frac{x}{\sqrt{5+4x-x^2}}dx = -\sqrt{5+4x-x^2}+2\sin^{-1}\left(\frac{x-2}{3}\right)+C$
3. $\displaystyle\int_{0}^{1} \frac{(1-x)}{\sqrt{8+2x-x^2}}dx = 3-2\sqrt{2}$
4. $\displaystyle\int_{-2}^{-1} \frac{x}{x^2+4x+5}dx = \ln\sqrt{2}-\frac{\pi}{2}$

5.7 The Integration of Rational Function_Partial Fractions:

We now come to the method of partial fraction, the basic technique for preparing rational functions for integration.

Example: two linear factor in the denominator, evaluate $\int \frac{5x-3}{(x+1)(x-3)}dx$

First: $\frac{5x-3}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$

We call A and B undetermined coefficients

$5x-3 = A(x-3) + B(x+1) = (A+B)x - 3A + B$

$x \Rightarrow 5 = A + B$

$x^0 \Rightarrow -3 = -3A + B$

$\therefore A = 2, and, B = 3$

$$\int \frac{5x-3}{(x+1)(x-3)}dx = \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx = 2\ln|x+1| + 3\ln|x-3| + C$$

Example: a repeated linear factor in the denominator, express $\frac{6x+7}{(x+2)^2}$ as a sum of partial fraction.

Solution:

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$6x+7 = A(x+2) + B = Ax + 2A + B$

$x \Rightarrow 6 = A$

$x^0 \Rightarrow 7 = 2A + B$

$\therefore A = 6, and, B = -5$

$$\therefore \frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}$$

Example: A quadratic factor in the denominator, solve, $\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$

Solution:

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$-2x+4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$

$-2x+4 = (A+B)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D)$

$x^3 \Rightarrow 0 = A + B$................................(1)

$x^2 \Rightarrow 0 = -2A + B - C + D$.............(2)

$x \Rightarrow -2 = A - 2B + C$.......................(3)

$x^0 \Rightarrow 4 = B - C + D$.........................(4)

$subtract, eq(4), and, eq(2) \Rightarrow A = 2$

$from, eq.(1) \Rightarrow B = 1, and, from, eq.(3) \Rightarrow C = -2, and, from, eq.(4) \Rightarrow D = 1$

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx = \int\left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx = \int\left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + C$$

Notes: for *f(x)/g(x)*:

1. let *(x-r)* be a linear function of *g(x)*. Suppose $(x-r)^m$ is the highest power of *(x-r)* that divides *g(x)*. Then the sum of m partial fraction to this factor as follows:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \ldots.. + \frac{A_m}{(x-r)^m}$$

2. let (x^2+px+q) be an irreducible quadratic factor of *g(x)*. Suppose $(x^2+px+q)^n$ is the highest power of this factor that divides *g(x),* the sum of the n partial fractions is:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \frac{B_3x + C_3}{(x^2+px+q)^3} + \ldots + \frac{B_nx + C_n}{(x^2+px+q)^n}$$

Example: Evaluate $\int \frac{x+4}{x^3+3x^2-10x}dx$

Solution:

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x^2+3x-10)} = \frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

$$x+4 = A(x-2)(x+5) + Bx(x+5) + Cx(x-2)$$

$$x+4 = Ax^2 + 5Ax - 2Ax - 10A + Bx^2 + 5Bx + Cx^2 - 2Cx$$

$$x+4 = (A+B+C)x^2 + (5B+3A-2C)x - 10A$$

$x^2 \Rightarrow 0 = A + B + C$.............(1)
$x \Rightarrow 1 = 5B + 3A - 2C$..........(2)
$x^0 \Rightarrow 4 = -10A$..................(3)

$\therefore A = -\frac{2}{5}, B = \frac{3}{7}, and, C = -\frac{1}{35}$

$\therefore \int \frac{x+4}{x^3 + 3x^2 - 10x} dx = -\frac{2}{5}\int \frac{dx}{x} + \frac{3}{7}\int \frac{dx}{x-2} - \frac{1}{35}\int \frac{dx}{x+5}$

$= -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C$

5.8 Integration by Rational Function by sinx and cosx (z integration method):

This method can be used when we have fraction contains sinx or cos x. we can use the following substitution:

$z = \tan\frac{x}{2}$

$\cos x = \frac{1-z^2}{1+z^2}$

$\sin x = \frac{2z}{1+z^2}$

$dx = \frac{2}{1+z^2}dz$

Example:

$$\int \frac{dx}{1+\cos x} = \int \frac{\frac{2dz}{1+z^2}}{1+\frac{1-z^2}{1+z^2}} = \int \frac{2}{2}dz = \int dz = z + C = \tan\frac{x}{2} + C$$

Example:

$$\int \frac{dx}{1-\sin x} = \int \frac{\frac{2dz}{1+z^2}}{1-\frac{2z}{1+z^2}} = \int \frac{\frac{2}{1+z^2}}{\frac{1+z^2-2z}{1+z^2}}dz = \int \frac{2}{1+z^2-2z}dz = \int \frac{2}{(z-1)^2}dz$$

$$= 2\int (z-1)^{-2}dz = \frac{-2}{(z-1)} + C = \frac{-2}{(\tan\frac{x}{2} - 1)} + C$$

H.W $\int \frac{dx}{2+\sin x}$

5.9 Integration by substituting u=z^n

This integration is used when root term present in integration:

Example:

$$\int \frac{x+2}{\sqrt{x+1}}dx$$

$$let \Rightarrow x+1=z^2 \Rightarrow z=\sqrt{x+1} \Rightarrow x=z^2-1 \Rightarrow dx=2zdz$$

$$\therefore \int \frac{(z^2-1)+2}{\sqrt{z^2}}(2zdz)=\int \frac{z^2+1}{z}.2zdz=2\int (z^2+1)dz$$

$$=2(\frac{z^3}{3}+z)+C=\frac{2}{3}(\sqrt{x+1})^3+2\sqrt{x+1}+C$$

Example:

$$\int x^3\sqrt{x^2+a^2}dx$$

$$let \Rightarrow z^2=x^2+a^2 \Rightarrow x^2=z^2-a^2 \Rightarrow x=\sqrt{z^2-a^2} \Rightarrow dx=\frac{2zdz}{2\sqrt{z^2-a^2}}$$

$$\therefore \int (\sqrt{z^2-a^2})^3.\sqrt{z^2}.\frac{zdz}{\sqrt{z^2-a^2}}=\int (z^2-a^2).z^2dz=\int (z^4-a^2z^2)dz$$

$$=\frac{z^5}{5}-\frac{a^2z^3}{3}+C=\frac{(\sqrt{x^2+a^2})^5}{5}-\frac{a^2(\sqrt{x^2+a^2})^3}{3}+C$$

Example:

$$\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}.dx$$

$$let \Rightarrow x=z^4 \Rightarrow z=\sqrt[4]{x} \Rightarrow dx=4z^3dz$$

$$\therefore \int \frac{\sqrt{z^4}}{1+\sqrt[4]{z^4}}.4.z^3dz=4\int \frac{z^2}{1+z}.z^3dz=4\int \frac{z^5}{1+z}.dz \Rightarrow \text{By long division} \Rightarrow$$

$$\begin{array}{r|l} & z^4-z^3+z^2-z+1 \\ \hline 1+z & z^5 \\ & \mp z^4 \mp z^5 \\ & -z^4 \\ & \pm z^3 \pm z^4 \\ & z^3 \\ & \mp z^2 \mp z^3 \\ & -z^2 \\ & \pm z \pm z^2 \\ & z \\ & \mp 1 \mp z \\ & -1 \end{array}$$

$$=4\int \frac{z^5}{1+z}dz=\int (z^4-z^3+z^2-z+1-\frac{1}{1+z})dz$$

$$=4(\frac{z^5}{5}-\frac{z^4}{4}+\frac{z^3}{3}-\frac{z^2}{2}+z-\ln|1+z|+C$$

$$=4(\frac{(\sqrt[4]{x})^5}{5}-\frac{(\sqrt[4]{x})^4}{4}+\frac{(\sqrt[4]{x})^3}{3}-\frac{(\sqrt[4]{x})^2}{2}+\sqrt[4]{x}-\ln|1+\sqrt[4]{x}|+C$$

Chapter Six

Hyperbolic Functions

6.1 Definition:

Some combination of e^x and e^{-x} are called hyperbolic functions. The hyperbolic sine and hyperbolic cosine are defined by following:

Hyperbolic sine of u:

$$\sinh u = \frac{e^u - e^{-u}}{2}$$

and,

$$\cosh u = \frac{e^u + e^{-u}}{2}$$

Note that:-

cosh u is often read "kosh u", and sinhu is often read "cinch u" or "shine u". Just as cosu and sinu may be identified with point (x,y) on the unit circle $x^2+y^2=1$, the function coshu and sinhu may be identified with the coordinate of pint (x,y) on the unit hyperbola $x^2-y^2=1$.

To check:

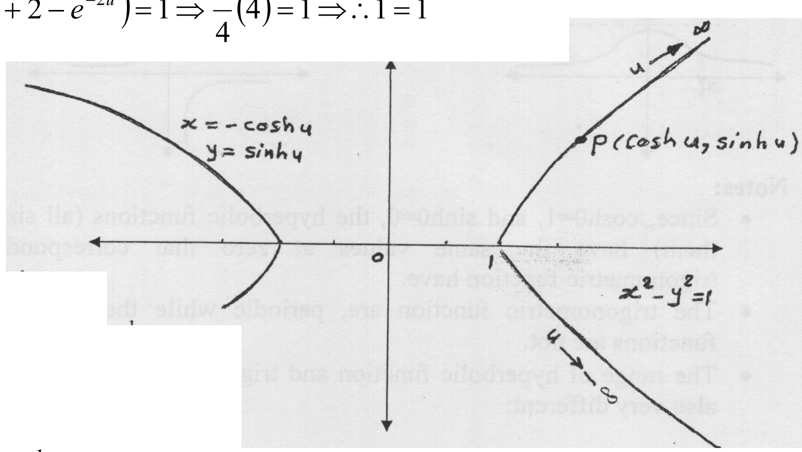
$$\left.\begin{array}{l} x = \cosh u \\ y = \sinh u \end{array}\right] \text{Lies on the unit hyperbola, then}$$

$$x^2 - y^2 = 1$$

$$\cosh^2 u - \sinh^2 u = 1$$

$$\frac{1}{4}\left(e^u + e^{-u}\right)^2 - \frac{1}{4}\left(e^u - e^{-u}\right)^2 = 1 \Rightarrow \frac{1}{4}\left(e^{2u} + 2 + e^{-2u}\right) - \frac{1}{4}\left(e^{2u} - 2 + e^{-2u}\right) = 1$$

$$\frac{1}{4}\left(e^{2u} + 2 + e^{-2u} - e^{2u} + 2 - e^{-2u}\right) = 1 \Rightarrow \frac{1}{4}(4) = 1 \Rightarrow \therefore 1 = 1$$



$$x = \cosh u = \frac{1}{2}\left(e^u + e^{-u}\right)$$

$$\because e^u, always, +ve, and, e^{-u} = \frac{1}{e^u}, also +ve$$

$$then, x, always +ve$$

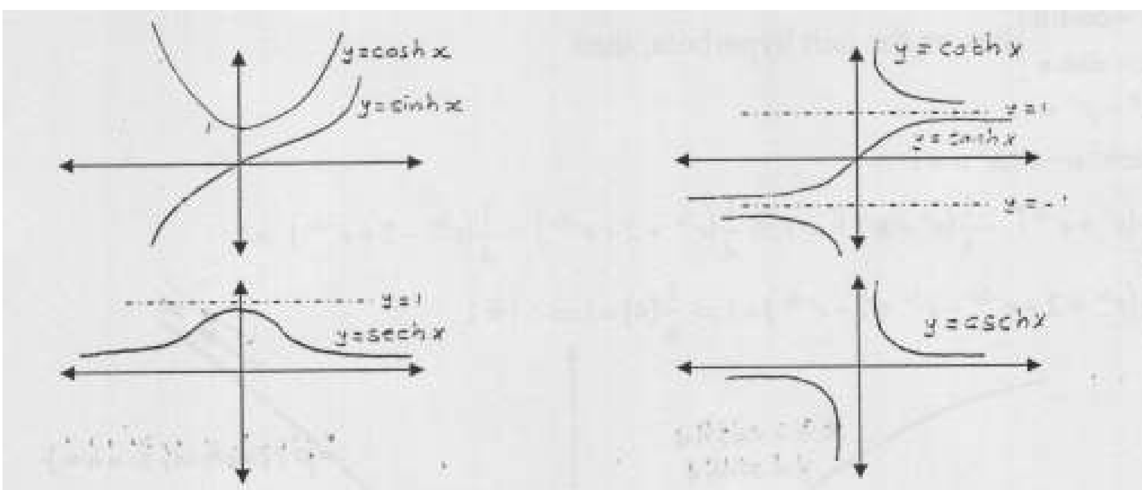
$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{e^{u} - e^{-u}}{e^{u} + e^{-u}}$$
$$\coth u = \frac{\cosh u}{\sinh u} = \frac{e^{u} + e^{-u}}{e^{u} - e^{-u}}$$
$$\sec hu = \frac{1}{\cosh u} = \frac{2}{e^{u} + e^{-u}}$$
$$\csc hu = \frac{1}{\sinh u} = \frac{2}{e^{u} - e^{-u}}$$

We have that:

$$\cosh^2 u - \sinh^2 u = 1$$
$$1 - \tanh^2 u = \sec h^2 u$$
$$\coth^2 u - 1 = \csc h^2 u$$
$$and,$$
$$\cosh u + \sinh u = e^{u}$$
$$\cosh u - \sinh u = e^{-u}$$

Thus, any combination of e^{u} and e^{-u} can be replaced a combination of *sinhu* and *coshu*.

The graphs of the six hyperbolic function are:



Notes:

- Since, cosh0=1, and sinh0=0, the hyperbolic functions (all six of them) have the same values at zero that corresponding trigonometric function have.

- The trigonometric functions are, periodic while the hyperbolic functions are not.
- The range of hyperbolic function and trigonometric functions are also very different:

sinx: oscillate between -1 and 1.

sinhx: increase steadily from -∞ to +∞.

cosx: oscillate between -1 and 1.

coshx: varies from +∞ to 1 to +∞ .

tanx: varies from -∞ to +∞.

tanhx: varies from -1 to +1.

Identities:

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$$
$$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$$
$$\sinh 2x = 2\sinh x\cosh x$$
$$\cosh 2x=\cosh^2 x+\sinh^2 x$$
$$\cosh^2 u=\frac{\cosh 2u+1}{2}$$
$$\sinh^2 u=\frac{\cosh 2u-1}{2}$$
$$\cosh^2 u-\sinh^2 u=1$$

Examples: write as simply as you can:

$$1.\ 2\cosh(\ln x)=2\left(\frac{e^{\ln x}+e^{-\ln x}}{2}\right)=x+\frac{1}{x}$$

$$2.\ \tanh(\ln x)=\frac{e^{\ln x}-e^{-\ln x}}{e^{\ln x}+e^{-\ln x}}=\frac{x-\frac{1}{x}}{x+\frac{1}{x}}=\frac{x^2-1}{x^2+1}$$

$$3.\ \cosh 5x+\sinh 5x=e^{5x}$$

$$4.\ \cosh 3x-\sinh 3x=e^{-3x}$$

$$5.\ \ln(\cosh x+\sinh x)+\ln(\cosh x-\sinh x)$$

$$\ln[(\cosh x+\sinh x)(\cosh x-\sinh x)]=\ln\left(\cosh^2 x-\sinh^2 x\right)=\ln(1)=0$$

6.2 *Derivative and Integration:*

Derivative:

$$\frac{d}{dx}(\sinh u) = \cosh u.\frac{du}{dx}$$
$$\frac{d}{dx}(\cosh u) = \sinh u.\frac{du}{dx}$$
$$\frac{d}{dx}(\tanh u) = \sec h^2 u.\frac{du}{dx}$$
$$\frac{d}{dx}(\coth u) = -\csc h^2 u.\frac{du}{dx}$$
$$\frac{d}{dx}(\sec hu) = -\sec hu \tanh u.\frac{du}{dx}$$
$$\frac{d}{dx}(\csc hu) = -\csc hu \coth u.\frac{du}{dx}$$

Integration:

$$\int \sinh u du = \cosh u + C$$
$$\int \cosh u du = \sinh u + C$$
$$\int \sec h^2 u du = \tanh u + C$$
$$\int \csc h^2 u du = -\coth u + C$$
$$\int \sec hu \tanh u du = -\sec hu + C$$
$$\int \csc hu \coth u du = -\csc hu + C$$

Example: $\int \coth 5x dx$

Solution:

$$\int \coth 5x dx = \frac{1}{5}\int \coth 5x.(5dx) = \frac{1}{5}\int \frac{\cosh 5x}{\sinh 5x} 5dx = \frac{1}{5}\ln\left|\sinh 5x\right| + C$$

Example: Evaluate $\int x \sinh x dx$

Solution: Integration by parts

$$\int u dv = uv - \int v du$$
$$u = x \Rightarrow du = dx$$
$$dv = \sinh x dx \Rightarrow v = \cosh x$$
$$\therefore \int x \sinh x dx = x \cosh x - \int \cosh x dx = x \cosh x - \sinh x + C$$

Example: Evaluate $\int_{0}^{\ln 4} e^x \sinh x dx$

Solution:

$$\int_0^{\ln 4} e^x \sinh x dx = \int_0^{\ln 4} e^x \frac{\left(e^x - e^{-x}\right)}{2} dx = \int_0^{\ln 4} \frac{\left(e^{2x} - 1\right)}{2} dx = \frac{1}{2}\left\{\frac{2}{2}\int_0^{\ln 4} e^{2x} dx - \int_0^{\ln 4} dx\right\}$$

$$= \left[\frac{e^{2x}}{4} - \frac{x}{2}\right]_0^{\ln 4} = \frac{e^{2\ln 4}}{4} - \frac{\ln 4}{2} - \frac{1}{4} = \frac{e^{\ln 16}}{4} - \frac{2\ln 2}{2} - \frac{1}{4} = 4 - \ln 2 - \frac{1}{4} = \frac{15}{4} - \ln 2$$

Examples: find dy/dx

$1. y = \sinh 3x$

$y' = 3\cosh 3x$

$2. y = \cosh^2 5x - \sinh^2 5x$

$note, that, \cosh^2 x - \sinh^2 x = 1 = cons\tan t$

$\therefore y' = 0$

$3. y = \coth(\tan x) \Rightarrow y' = -\csc h^2(\tan x).\sec^2 x$

$4. y = \sec h^2 x + \tanh^2 x$

$y' = 2\sec hx.(-\sec hx \tanh xdx) + 2\tanh x.(\sec h^2 xdx)$

$= -2\sec h^2 x \tanh xd + 2sexh^2 x \tanh x = 0$

$5. y = \sin^{-1}(\tanh x)$

$$y' = \frac{1}{\sqrt{1 - \tanh^2 x}} \sec h^2 xdx = \frac{\sec h^2 x}{\sec hx} dx = \sec hxdx$$

$$6. \sinh y = \tan x \Rightarrow \cosh y.y' = \sec^2 x \Rightarrow y' = \frac{\sec^2 x}{\cosh y}$$

Example: Evaluate

$$1. \int \frac{\sinh x}{\cosh^4 x} dx = \int \cosh^{-4} x.\sinh x.dx = -\frac{1}{3}\cosh^{-3} x + C = -\frac{1}{3}\sec h^3 x + C$$

$$2. \int \frac{\sinh x}{(1 + \cosh x)} dx = \ln(1 + \cosh x) + C$$

$$3. \int_0^{\ln 2} \frac{1 - e^{-2x}}{1 + e^{-2x}} dx. \times \frac{e^x}{e^x} = \int_0^{\ln 2} \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int_0^{\ln 2} \tanh xdx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x} dx$$

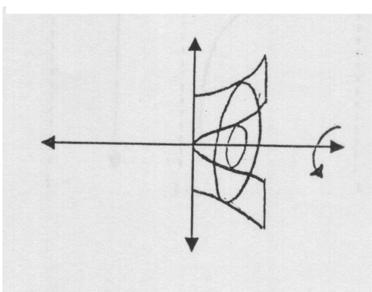
$$= [\ln(\cosh x)]_0^{\ln 2} = \ln(\cosh \ln 2) - \ln 1 = \ln\frac{5}{4}$$

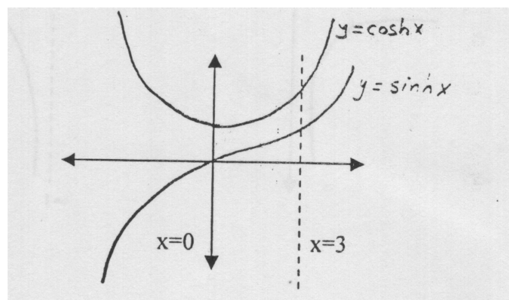
4. Use tabular integration to evaluate $\int x^2 \cosh x dx$.

f(x)		g(x)
x^2	+	coshx
2x	-	sinhx
2	+	coshx
0		sinhx

$$\therefore \int x^2 \cosh x dx = x^2 \sinh x - 2x \cosh x + 2\sinh x + C$$

Example: the region between the curve y=sinhx, and y=coshx, and the lines x=0 and x=3, is revolved about the x-axis to generate a solid. Find the volume of the solid.





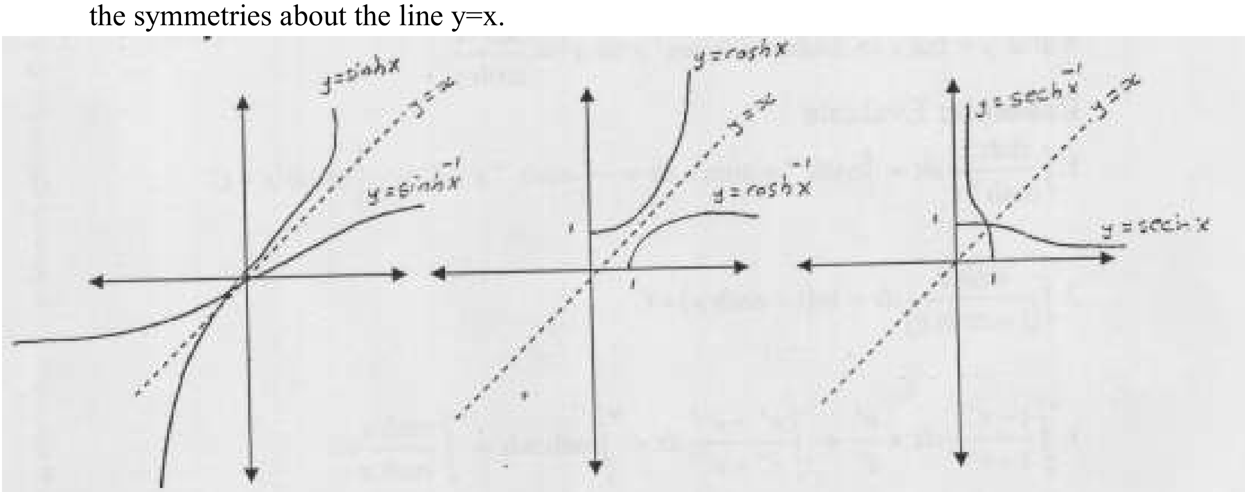
Solution: the volume can be obtained using washer formula

$$V = \int_a^b \pi\left[R^2(x) - r^2(x)\right]dx$$

$$= \int_0^3 \pi\left(\cosh^2 x - \sinh^2 x\right)dx = \int_0^3 \pi(1)dx = [\pi x]_0^3 = 3\pi$$

6.3 Inverse Hyperbolic Function:

The graphs of $y = \sinh^{-1} x, y = \cosh^{-1} x, and, y = \operatorname{sec}h^{-1} x,$ are shown below. Note the symmetries about the line y=x.



$y=\sinh^{-1}x$ " is read y equal the inverse hyperbolic sine of x".

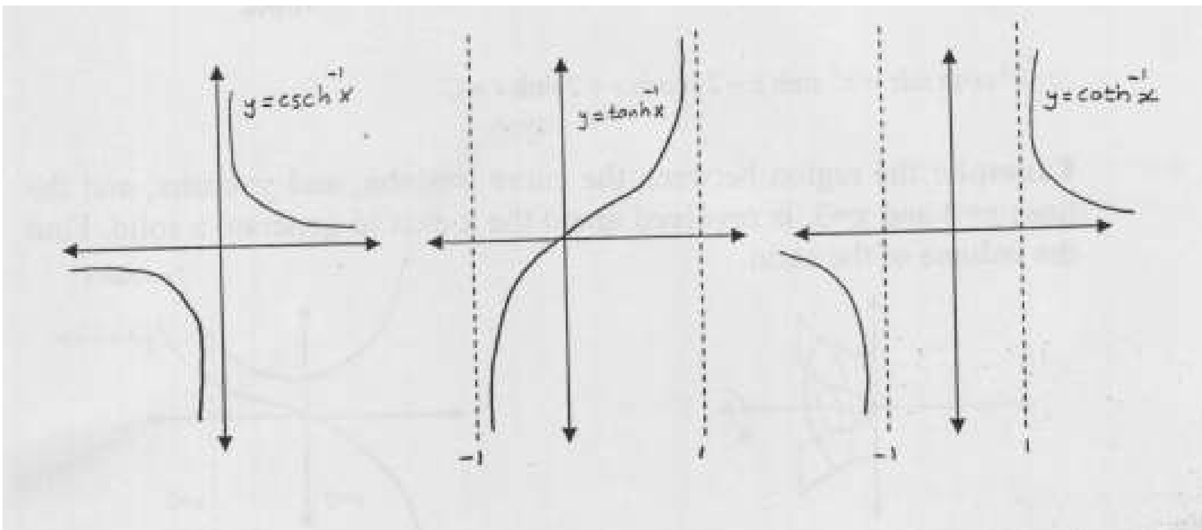
Domain of:

$y=\sinh^{-1}x$, $-\infty< x <\infty$.

$y=\cosh^{-1}x$, $x \geq 1$

$y=\text{sech}^{-1}x$ x interval (0,1].

The other inverse hyperbolic functions are:



Useful Identities:

The inverse hyperbolic secant, cosecant, and cotangent satisfy the identities:

$$\sec h^{-1}x = \cosh^{-1}\frac{1}{x}$$

$$\csc h^{-1}x = \sinh^{-1}\frac{1}{x}$$

$$\coth^{-1}x = \tanh^{-1}\frac{1}{x}$$

Logarithmic Formulas for Evaluating Inverse Hyperbolic Functions:

- $\sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$ $-\infty< x <\infty$.
- $\cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right)$ $x \geq 1$.
- $\tanh^{-1}x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$ $|x| < 1$.
- $\sec h^{-1}x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right) = \cosh^{-1}\left(\dfrac{1}{x}\right)$ $0< x \leq 1$.

- $\csc h^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) = \sinh^{-1}\left(\frac{1}{x}\right)$ x ≠ 0.
- $\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1} = \tanh^{-1}\left(\frac{1}{x}\right)$ | x | > 1.

Above are a conversion formula, when hyperbolic function keys are not available on a calculator.

Example:

$$\tanh^{-1}\left(\frac{1}{4}\right) = \frac{1}{2}\ln\frac{1+\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{2}\ln\frac{\frac{5}{4}}{\frac{3}{4}} = \frac{1}{2}\ln\frac{5}{3} = \frac{1}{2}(\ln 5 - \ln 3) = 0.25541$$

Example: Derive the formula $\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$.

Solution:

$$let, y = \tanh^{-1} x, then, \Rightarrow x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} = \frac{e^y - \frac{1}{e^y}}{e^y + \frac{1}{e^y}}$$

$$x = \frac{e^{2y} - 1}{e^{2y} + 1} \Rightarrow xe^{2y} + x = e^{2y} - 1 \Rightarrow 1 + x = e^{2y}(1-x) \Rightarrow e^{2y} = \left(\frac{1+x}{1-x}\right)$$

$$2y = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tanh^{-1} x$$

The other formula can be derived in a similar way.

Derivative and Integration:

Derivative:

$$\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{1}{\sqrt{1+u^2}}.\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cosh^{-1} u\right) = \frac{1}{\sqrt{u^2-1}}.\frac{du}{dx} \Rightarrow u\rangle 1$$

$$\frac{d}{dx}\left(\tanh^{-1} u\right) = \frac{1}{1-u^2}.\frac{du}{dx} \Rightarrow |u|\langle 1$$

$$\frac{d}{dx}\left(\coth^{-1} u\right) = \frac{1}{1-u^2}.\frac{du}{dx} \Rightarrow |u|\rangle 1$$

$$\frac{d}{dx}\left(\sec h^{-1} u\right) = \frac{-1}{u\sqrt{1-u^2}}.\frac{du}{dx} \Rightarrow 0\langle u\langle 1$$

$$\frac{d}{dx}\left(\csc h^{-1} u\right) = \frac{-1}{|u|\sqrt{1+u^2}}.\frac{du}{dx} \Rightarrow u \neq 0$$

Integration:

$$\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + C \Rightarrow |u|\langle 1 \\ \coth^{-1} u + C \Rightarrow |u|\rangle 1 \end{cases} = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\sec h^{-1}|u| + C = -\cosh^{-1}\left(\frac{1}{|u|}\right) + C$$

$$\int \frac{du}{u\sqrt{1+u^2}} = -\csc h^{-1}|u| + C = -\sinh^{-1}\left(\frac{1}{|u|}\right) + C$$

Example: evaluate $\int_0^{\frac{1}{2}} 2x \tanh^{-1} x dx$

Solution: integration by parts

$$\int u dv = uv - \int v du$$

$$let, u = \tanh^{-1} x \Rightarrow du = \frac{1}{1-x^2} dx$$

$$dv = 2x dx \Rightarrow v = x^2$$

$$\therefore \int 2x \tanh^{-1} x dx = x^2 \tanh^{-1} x - \underbrace{\int \frac{x^2}{1-x^2} dx}_{long, division}$$

$$\frac{x^2}{1-x^2} = 1 + \underbrace{\frac{1}{1-x^2}}_{Partial, fraction}$$

$$\begin{array}{r} 1 \\ x^2-1 \overline{\smash{)}\, x^2} \\ \underline{\pm x^2 \pm 1} \\ 1 \end{array}$$

$$\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}$$

$$1=A(x-1)+B(x+1)\Rightarrow Ax-A+Bx+B=1$$

$$\therefore B=\frac{1}{2}, and, A=-\frac{1}{2}$$

$$\therefore \frac{x^2}{x^2-1}=1+\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$$

$$\therefore \int 2x\tanh^{-1}x dx=x^2\tanh^{-1}x+\int\left(1+\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\right)dx$$

$$=x^2\tanh^{-1}x+x+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C=x^2\tanh^{-1}x+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$$

$$For, limits, of, integration:$$

$$\int_0^{\frac{1}{2}}2x\tanh^{-1}x dx=\left[\frac{x^2}{2}\ln\frac{1+x}{1-x}+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_0^{\frac{1}{2}}=\frac{1}{8}\ln\frac{3/2}{1/2}+\frac{1}{2}+\frac{1}{2}\ln\frac{1/2}{3/2}$$

$$=\frac{1}{8}\ln 3+\frac{1}{2}+\frac{1}{2}\ln\frac{1}{3}=\frac{1}{8}\ln 3+\frac{1}{2}-\frac{1}{2}\ln 3=\left(\frac{1}{8}-\frac{1}{2}\right)\ln 3+\frac{1}{2}=\frac{1}{2}-\frac{3}{8}\ln 3$$

Example: show that $\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$

Solution:

$$let, y=\sinh^{-1}x\Rightarrow x=\sinh y=\frac{\left(e^y-e^{-y}\right)}{2}$$

$$2x=e^y-e^{-y}\Rightarrow 2x=e^{-y}\left(e^{2y}-1\right)\Rightarrow e^{2y}-1=2xe^y\Rightarrow e^{2y}-2xe^y-1=0$$

$$solve, by\Rightarrow az^2+bz+c=0\Rightarrow\therefore z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore \left(e^y\right)^2-2x\left(e^y\right)-1=0\Rightarrow\therefore a=1, b=-2x, and, c=-1$$

$$\therefore e^y=\frac{2x\pm\sqrt{4x^2+4}}{2}=\frac{2x\pm 2\sqrt{x^2+1}}{2}=x\pm\sqrt{x^2+1}$$

$$\because e^y, always+ve$$

$$\therefore e^y=x+\sqrt{x^2+1}\Rightarrow y=\ln\left[x+\sqrt{x^2+1}\right]=\sinh^{-1}x$$

Problems:

1. If *sinhx=tany*, show that, *x=ln(secy+tany).*
2. If *a=c coshx*, and, *b=c sinhx*, prove that $(a+b)^2 e^{-2x} = a^2 - b^2$.
3. Show that, $\tanh^{-1}\left(\frac{x^2-1}{x^2+1}\right) = \ln x$.
4. Solve for real value of x, *3cosh2x=3+sinh2x*.
5. Prove that, $\frac{1+\tanh x}{1-\tanh x} = e^{2x}$.
6. If, $t = \tanh\left(\frac{x}{2}\right)$, prove that, $\sinh x = \frac{2t}{1-t^2}, and, \cosh x = \frac{1+t^2}{1-t^2}$. Then solve the equation, $7\sinh x + 20\cosh x = 24$.

Chapter Seven

Matrices

7.1 Determinants:

A rectangular array of numbers like:

$$A=\begin{bmatrix}2 & 1 & 3\\ 1 & 0 & -2\end{bmatrix}$$

Is called a matrix. We call A, a "2 by 3" matrix because it has two rows and three columns. An "m by n" matrix has m rows and n columns, and entry or elements (numbers) in the i^{th} row and j^{th} column is often denoted by a_{ij}.

$$\begin{bmatrix} & & & j^{th}\,column & & \\ & & & . & & \\ & & & . & & \\ i^{th}\,row & . & . & a_{ij} & . & . \\ & & & . & & \\ & & & . & & \end{bmatrix}$$

The matrix:

$$A=\begin{bmatrix}2 & 1 & 3\\ 1 & 0 & -2\end{bmatrix}$$

Has, $a_{11}=2$ $a_{12}=1$ $a_{13}=3$ $a_{21}=1$ $a_{22}=0$ $a_{23}=-2$.

A matrix with the same number of rows as column is a *square matrix*. It is a matrix of order n if the number of rows and column is n. With each square matrix A, we associate a number det A, or $|a_{ij}|$, called the determinant of A, calculated from the elements of A.

$$For, n=1$$
$$\det[a]=a$$
$$For, n=2$$
$$\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=a_{11}a_{22}-a_{21}a_{12}$$
$$For, n=3$$
$$\det\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}$$

Example: Find the determinant of the matrix A:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution:

$$\det A = 2\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} = 2(-1+6) - 1(3+4) + 3(9+2) = 10 - 7 + 33 = 36$$

Useful Facts about Determinants:

1. If two rows of a matrix are identical, the determinant is zero.
2. Interchanging two rows of a matrix changes the sign of its determinant.
3. If each element of same row or column of a matrix is multiplied by a constant c, the determinant is multiplied by c.
4. If all elements of a matrix above the main diagonal (or below it) are zero, the determinant of a matrix is product of the element of the main diagonal, for example:

$$\begin{vmatrix} 3 & 4 & 7 \\ 0 & -2 & 5 \\ 0 & 0 & 5 \end{vmatrix} = (3)(-2)(5) = -30$$

Main diagonal

7.2 Cramer's Rule:

For the system of equation:

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$
$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

If D≠0, the above system has unique solution, and Cramer's rule state that:

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D}$$
$$y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

Example: Solve the system

$$3x - y = 9$$
$$x + 2y = -4$$

Solution: The determinant of a matrix is:

$$D = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 6 + 1 = 7$$

$$x = \frac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{D} = \frac{18 - 4}{7} = \frac{14}{7} = 2$$

$$y = \frac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{D} = \frac{-12 - 9}{7} = \frac{-21}{7} = -3$$

Example: Solve the system of equations:

$$2x_1 + x_2 - x_3 = 4$$
$$x_1 - 2x_2 + x_3 = -10$$
$$-3x_1 - 2x_3 = 9$$

Solution: First the matrix A and B:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ -3 & 0 & -2 \end{bmatrix}, and, B = \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}$$

Then, find the determinant of A:

$$\det A = 2\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -3 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -2 \\ -3 & 0 \end{vmatrix} = 2(4-0) - 1(-2+3) - 1(0-6) = 13$$

$$x_1 = \frac{\begin{vmatrix} 4 & 1 & -1 \\ -10 & -2 & 1 \\ 9 & 0 & -2 \end{vmatrix}}{13} = \frac{4\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} - 1\begin{vmatrix} -10 & 1 \\ 9 & -2 \end{vmatrix} - 1\begin{vmatrix} -10 & -2 \\ 9 & 0 \end{vmatrix}}{13} = \frac{4(4) - 1(11) - 1(18)}{13} = -1$$

$$x_2 = \frac{\begin{vmatrix} 2 & 4 & -1 \\ 1 & -10 & 1 \\ -3 & 9 & -2 \end{vmatrix}}{13} = \frac{2\begin{vmatrix} -10 & 1 \\ 9 & -2 \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ -3 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -10 \\ -3 & 9 \end{vmatrix}}{13} = \frac{22 - 4 + 21}{13} = 3$$

$$x_3 = \frac{\begin{vmatrix} 2 & 1 & 4 \\ 1 & -2 & -10 \\ -3 & 0 & 9 \end{vmatrix}}{13} = \frac{2\begin{vmatrix} -2 & -10 \\ 0 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & -10 \\ -3 & 9 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ -3 & 0 \end{vmatrix}}{13} = \frac{2(-18) - 1(-21) + 4(-6)}{13} = -3$$

Example: Use Cramer's rule to solve the system:

$$3\tan^2 x + y = 3$$
$$2\tan^2 x - 3y = 13$$

Solution: Let u=tan^{2}x, then the system becomes:

$3u + y = 3$

$2u - 3y = 13$

$the, matrix, A = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}, and, B = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$

$\therefore \det A = -9 - 2 = -11$

$u = \dfrac{\begin{vmatrix} 3 & 1 \\ 13 & -3 \end{vmatrix}}{-11} = \dfrac{-9-13}{-11} = \dfrac{-22}{-11} = 2 \Rightarrow \therefore \tan^2 x = 2 \Rightarrow \tan x = \sqrt{2} \Rightarrow \therefore x = \tan^{-1}\sqrt{2}$

$y = \dfrac{\begin{vmatrix} 3 & 3 \\ 2 & 13 \end{vmatrix}}{-11} = \dfrac{39-6}{-11} = \dfrac{33}{-11} = -3$

H.W

1. Solve the system of equation

$3\tan^2 x + y = 3$

$2\sec^2 x - 3y = 13$

2. Find the values of x, y and z from the system of equations:

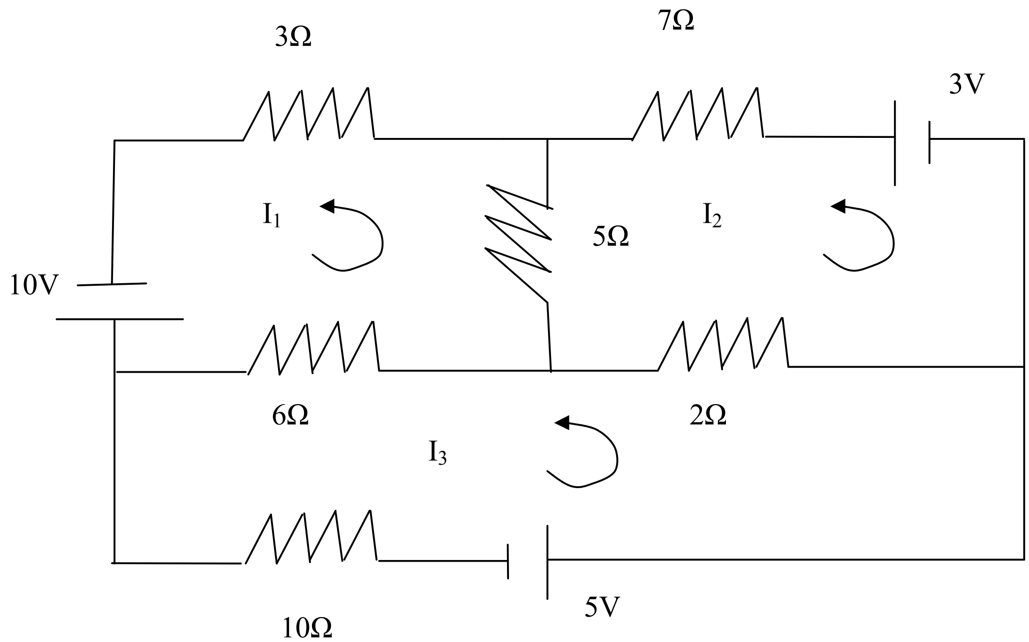
$\tan^2 z - \dfrac{1}{y^2} = -1$

$x^2 + \dfrac{3}{y^2} = 7$

$-y^2 \sin^2 z + \left(2 - 3y^2\right)\cos^2 z = 0$

Example: Application

For a simple electrical circuit



Where I in amperes, V in volts and R in ohm.

Solution: From the basic Kirchoff`s laws:

- For each closed circuit the total algebraic sum of e.m.f is zero.
- At each nodal point there is continuity of current.

$$10 = 6(I_1 - I_3) + 5(I_1 - I_2) + 3I_1$$
$$3 = 7I_2 + 5(I_2 - I_1) + 2(I_2 - I_3)$$
$$5 = 2(I_3 - I_2) + 6(I_3 - I_1) + 10I_3$$

The set of equation reduced to the following:

$$14I_1 - 5I_2 - 6I_3 = 10$$
$$-5I_1 + 14I_2 - 2I_3 = 3$$
$$-6I_1 - 2I_2 + 18I_3 = 5$$

Then, the matrix $A = \begin{bmatrix} 14 & -5 & -6 \\ -5 & 14 & -2 \\ -6 & -2 & 18 \end{bmatrix}, and, B = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}$

We now can find each current I_1, I_2, *and* I_3 by Cramer's Rule.

$$D = \det A = 14\begin{vmatrix} 14 & -2 \\ -2 & 18 \end{vmatrix} - (-5)\begin{vmatrix} -5 & -2 \\ -6 & 18 \end{vmatrix} + (-6)\begin{vmatrix} -5 & 14 \\ -6 & -2 \end{vmatrix} = 2398$$

$$\therefore I_1 = \frac{\begin{bmatrix} 10 & -5 & -6 \\ 3 & 14 & -2 \\ 5 & -2 & 18 \end{bmatrix}}{D} = \frac{10\begin{vmatrix} 14 & -2 \\ -2 & 18 \end{vmatrix} - (-5)\begin{vmatrix} 3 & -2 \\ 5 & 18 \end{vmatrix} + (-6)\begin{vmatrix} 3 & 14 \\ 5 & -2 \end{vmatrix}}{2398} = \frac{2840 + 320 + 456}{2398} = 1.5A$$

In the same way we can calculate the values of other currents, then:

$I_2 = 0.81A, and, I_3 = 0.82A$.

7.3 Minors and Cofactors:

For the formula:

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The second order determinant on the right-hand side of above equation is called the minors.

$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ is the minor of a_{11}

$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ is the minor of a_{12}

Generally:

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ The minor of a_{22} is $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ The minor of a_{23} is $\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

The cofactor of a_{ij} is the determinant A_{ij} that is $(-1)^{i+j}$ times the minor of a_{ij}.

$$A_{22} = (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

7.4 Transpose of a Matrix:

The determinant of transpose of a matrix is to write the rows as column, for example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, Matrix, Transpose \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

7.5 Matrix Algebra:

1. Equality of Matrices: Two matrices are equal if and only if they are of the same order and their elements are equal.
2. Addition: Consider the two matrices;

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, and, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Then,

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

- Note that no meaning can be attached to addition of matrices of different order.
- $A + B = B + A$
- $(A + B) + C = A + (B + C)$.

Example: $$\begin{bmatrix} 3 & 6 \\ 7 & 2 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -1 & 5 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 6 & 7 \\ 8 & -2 \end{bmatrix}$$

3. Multiplication by Scalar: If a matrix is multiplied by a scalar then every element is multiplied by the same scalar.

Examples:

$$3\begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 3 \\ 12 & 9 & 0 \end{bmatrix}$$

$$-1\begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -5 \end{bmatrix}$$

4. Zero Matrix: The subtraction of two equal matrices of order m×n gives a zero matrix of order m×n.

Example: $\begin{bmatrix} 18 & 13 & 8 \\ 8 & 13 & -12 \end{bmatrix} - \begin{bmatrix} 18 & 13 & 8 \\ 8 & 13 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. Multiplication of Matrices: Given a matrix A of order m×n and a second matrix B of order n×p, the product AB is a third matrix C of order m×p.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, and, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\therefore AB = C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

Example: $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 4 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10+12+3 & 6+0+1 \\ -5+8+12 & -3+0+4 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 15 & 1 \end{bmatrix}$

(2×3) (3×2) (2×2)

matrix

Example: $\begin{bmatrix} 5 & 3 \\ 4 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 10-3 & 15+6 & 5+12 \\ 8+0 & 12+0 & 4+0 \\ 6-1 & 9+2 & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & 21 & 17 \\ 8 & 12 & 4 \\ 5 & 11 & 7 \end{bmatrix}$

(3×2) (2×3) (3×3)

matrix

Note that for the product of AB to exist it is essential that <u>*there are the same number of columns in A as there is rows in B*</u>.

Example: $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 5 & 6 & 7 \end{bmatrix}$ "has no meaning"

(2×2) (1×3)

- We can not product the above matrices, because number of columns in the first matrix <u>not equals</u> the number of rows in the second matrix.
- Note that, also, AB≠BA.

Examples:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

6. Unit Matrix: It is matrix whose elements in the main diagonal are all equal one, and denoted by I_n, for example:

$$I_1 = [1]$$
$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Evaluate *IA* and *AI* when, $A = \begin{bmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}$,and *I* is a unit matrix.

Solution:

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}$$

And, $AI = \begin{bmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}$

We can see that $I_nA=A$

H.W

If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, then, show, that, A^2 - 5A + 4I = 0$

7. The Inverse Matrix: To find the inverse of matrix whose determinant is not zero:

a- Construct the matrix of cofactors of A

$$cofA = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

b- Construct the transposed matrix of cofactors (called the adjiont of A)

$$adjA = (cofA)^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

"Transpose, mean write the rows as columns"

c- Then, the inverse is:

$$A^{-1} = \frac{adjA}{\det A}$$

Example: Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$$

Solution: Check the determinant:

$$\det A = 2\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + (-4)\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 2(-2+3) - 3(-1-9) - 4(-1-6) = 2 + 30 + 28 = 60$$

since, the determinant not zero, then the matrix have inverse.

First find the matrix of cofactors of A:

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} = (-1)^2(-2+3) = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} = -1(-1-9) = 10$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1(-1-6) = -7$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & -4 \\ -1 & -1 \end{vmatrix} = -1(-3-4) = 7$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & -4 \\ 3 & -1 \end{vmatrix} = (-1)^4(-2+12) = 10$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -1(-2-9) = 11$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} = 1(9+8) = 17$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & -4 \\ 1 & 3 \end{vmatrix} = -1(6+4) = -10$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1(4-3) = 1$$

$$cofA = \begin{bmatrix} 1 & 10 & -7 \\ 7 & 10 & 11 \\ 17 & -10 & 1 \end{bmatrix}$$

$$adjA = (cofA)^T = \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adjA}{\det A} = \frac{1}{60}\begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$$

Note that: $AA^{-1} = A^{-1}A = I$

H.W

1. Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 8 & 9 \\ 0 & 4 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

2. Solve the following system of equation:

$$x + 8y + 9z = 10$$
$$4y + 6z = 10$$
$$3z = -10$$

Answer :

$$1.A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & \frac{1}{4} & \frac{-1}{2} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$2.x = -20, y = \frac{15}{2}, z = -\frac{10}{3}$$

Chapter Eight

Complex Numbers

10.1 Introduction:

Complex numbers are expression of the form (a+ib), where a and b are real number and, i is a symbol for $\sqrt{-1}$.

1. In the system of all integer, we can solve all equations of the form $x+a=0$, where (a) may be any integer.
2. In the system of all rational numbers, we can solve all equations of the form $ax+b=0$, $a\neq 0$.
3. In the system of the form $ax^2+bx+c=0$, having $a\neq 0$, and $b^2-4ac\geq 0$, we can solve this equation by:

$$x=\frac{-b\mp\sqrt{b^2-4ac}}{2a} \qquad \dots (1)$$

If the discriminate, $d=b^2-4ac$ is negative, the solution of equation (1) do not belong to any of the system discussed above. In fact, the very simple quadratic equation $x^2+1=0$ is impossible to solve by any way discussed above. Thus, the fourth invented system, the set of complex numbers (a+ib), we call, ***a, the real part***, and ***b, the imaginary part***.

10.2 Definitions:

1. Equality: a+ib=c+id, if and only if, a=c and b=d.
2. Addition: (a+ib)+(c+id)=(a+c)+i(b+d).
3. Subtraction: (a+ib)-(c+id)=(a-c)+i(b-d).
4. Multiplication:(a+ib).(c+id)=(ac-bd)+i(ad+bc). And, c(a+ib)=ac+ibc.
5. Division: To reduce any rational combination of complex numbers to a single complex number, we apply the laws of elementary algebra, replacing i^2 by -1.

$$\frac{c+id}{a+ib}=\frac{(c+id)(a-ib)}{(a+ib)(a-ib)}=\frac{(ac+bd)+i(ad-bc)}{a^2+b^2}$$

The result is a complex number X+iY, with $X=\frac{ac+bd}{a^2+b^2}, and, Y=\frac{ad-bc}{a^2+b^2}$.

The number a-ib is called the *Complex Conjugate* of a+ib, it denoted by $\bar{z}$.

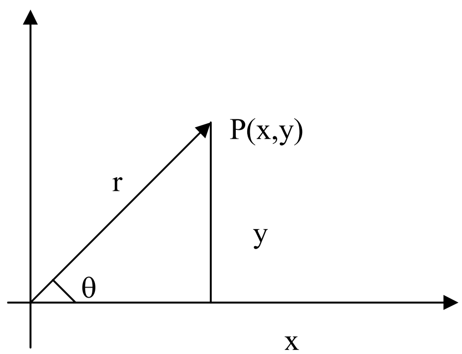
$\therefore \bar{z} = a - ib$, $\Rightarrow$ The complex conjugate of denominator.

10.3 Argand Diagram:

There are two geometric representation of the complex number z=x+iy:

a- As the point p(x,y) in the xy-plane.

b- As the vector $\overrightarrow{OP}$ from origin to P.



Argand Diagram

x-axis is real axis

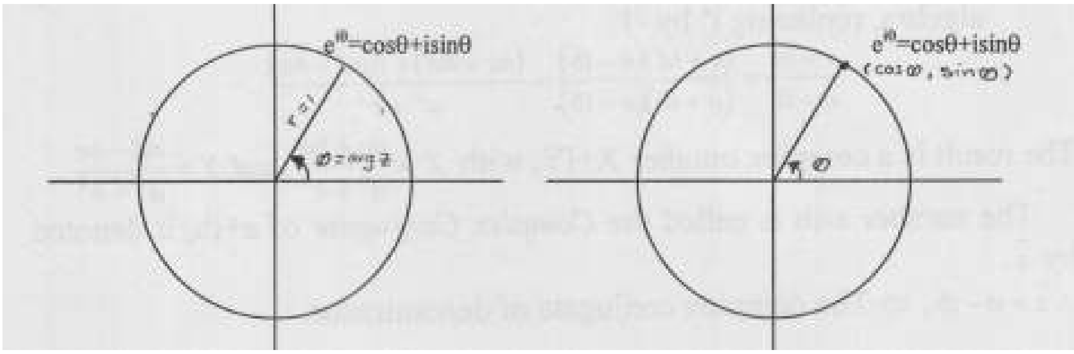
y-axis is imaginary axis.

- In term of polar coordinate of x and y we have, $x = r\cos\theta, and, y = r\sin\theta$, and, $z = x + iy = r(\cos\theta + i\sin\theta)$.
- Absolute value of complex number x+iy is the length r of vector $\overrightarrow{OP}$ from origin to P(x,y), $\therefore |x + iy| = \sqrt{x^2 + y^2} = r$.
- Polar angle θ is called *argument of z* and written as $\theta = \arg z$.

Useful Identity:

1. $e^{i\theta} = \cos\theta + i\sin\theta$
2. $e^{i\theta_1} . e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$
3. $i^2 = -1, i^3 = i^2 i = -i, i^4 = i^2 i^2 = 1, ...etc$
4. $z.\bar{z} = |z|^2$

10.4 Product, Quotient, Power and Roots:



$e^{i\theta}=\cos\theta+i\sin\theta$ $e^{i\theta}=\cos\theta+i\sin\theta$

Argand diagram for $e^{i\theta}$ as a vector Argand diagram for $e^{i\theta}$ as a point

Products: To multiply two complex number, in polar coordinates, we multiply their absolute value and add their angles. Let:

$$z_1 = r_1 e^{i\theta_1}, and, z_2 = r_2 e^{i\theta_2}$$
$$|z_1| = r_1, \Rightarrow \arg z_1 = \theta_1, and, |z_1| = r_2 \Rightarrow \arg z_2 = \theta_2$$
$$Then,$$
$$z_1 z_2 = r_1 e^{i\theta_1}.r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$
$$|z_1 z_2| = r_1 r_2 = |z_1|.|z_2|, and, \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

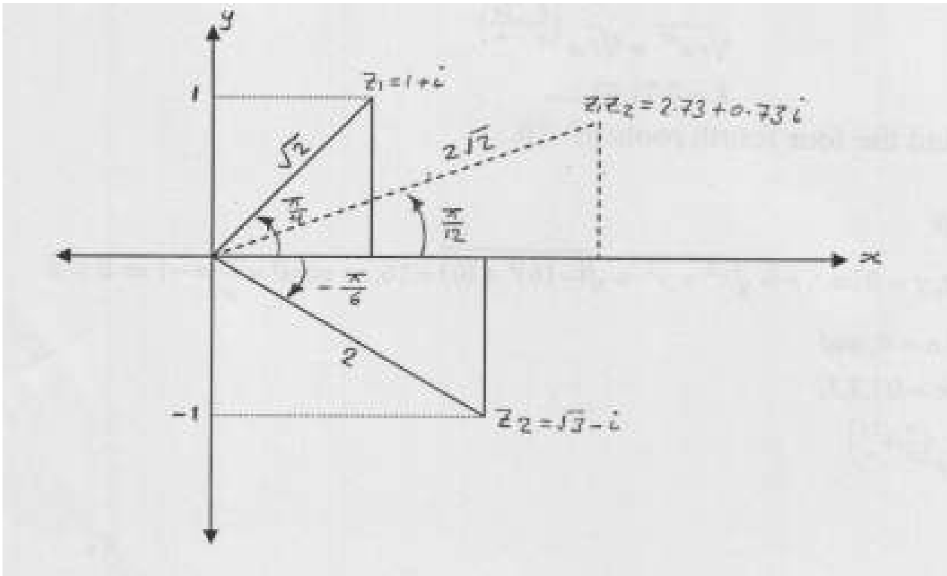
Example: Let $z_1 = 1 + i, and, z_2 = \sqrt{3} - i$, plot these two complex number in Argand diagram and find their products.

Solution: generally, z=x+iy, then:

For z_1:	*For z_2:*
$x_1 = 1$ $y_1 = 1$ $r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{2}$ $x_1 = r_1 \cos\theta_1$ $\cos\theta_1 = \frac{x_1}{r_1} = \frac{1}{\sqrt{2}}$ $\therefore \theta_1 = \frac{\pi}{4}$	$x_2 = \sqrt{3}$ $y_2 = -1$ $r_2 = \sqrt{x_2^2 + y_2^2} = 2$ $x_2 = r_2 \cos\theta_2$ $\cos\theta_2 = \frac{x_2}{r_2} = \frac{\sqrt{3}}{2}$ $\therefore \theta_2 = -\frac{\pi}{6}$

$$\therefore z_1 z_2 = r_1 r_2 . e^{i(\theta_1+\theta_2)} = 2\sqrt{2}.e^{\left(i\frac{\pi}{4} - i\frac{\pi}{6}\right)} = 2\sqrt{2}.e^{\left(i\frac{\pi}{12}\right)} = 2\sqrt{2}.\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \cong (2.73 + 0.73i)$$

The product of two complex number is represented by a vector whose length is the product of the length of two factors $\left|z_1 z_2\right| = r_1 r_2 = 2\sqrt{2}$.



Quotient: Suppose $r_2 \neq 0$, then,

$$\frac{z_1}{z_2} = \frac{r_1.e^{i\vartheta_1}}{r_2.e^{i\theta_2}} = \frac{r_1}{r_2}.e^{i(\theta_1-\theta_2)}$$

$$and,$$

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

That is, we divided the lengths and subtract angles.

Example: Let $z_1 = 1+i, and, z_2 = \sqrt{3} - i$. Find the division.

Solution:

$$\frac{z_1}{z_2} = \frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}.e^{i\frac{\pi}{4}}}{2.e^{-i\frac{\pi}{6}}} = \frac{\sqrt{2}}{2}.e^{i\frac{5\pi}{12}} = 0.707\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) = 0.183 + 0.683i$$

Powers: If n is a positive integer, then, $z^n = z.z.z....z$, (n factors). For $z = re^{i\theta}$, we obtained:

$$\boxed{z^n = \left(r.e^{i\theta}\right)^n = r^n.e^{in\theta}} \qquad \ldots \qquad (2)$$

If we place r=1 in equation (2), we obtain the *DeMoiver`s Theorem*:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta} \qquad \dots \qquad (3)$$

For example, if n=3, then, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$.

Roots: If $z=re^{i\theta}$, is a complex number different from zero, and n is a positive integer, then there are different complex number $w_0, w_1, w_2, \dots w_{n-1}$, that are the n^{th} roots of z. All the n^{th} roots of $z=re^{i\theta}$, are given by:

$$\sqrt[n]{r.e^{i\theta}} = \sqrt[n]{r}.e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)}$$
$$k = 0, \mp 1, \mp 2, \dots..$$

Example: Find the four fourth roots of -16.

Solution:

$$z = -16 = x + iy$$

$$\therefore x = -16, and, y = 0 \Rightarrow \therefore r = \sqrt{x^2 + y^2} = \sqrt{(-16)^2 + (0)} = 16, \Rightarrow \cos\theta = \frac{x}{r} = -1 \Rightarrow \theta = \pi$$

$$fourth, root, \therefore n = 4, and$$
$$four, roots, \therefore k = 0,1,2,3.$$
$$\therefore \sqrt[n]{re^{i\theta}} = \sqrt[n]{r}.e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)}$$

And the four roots is:

$at, \underline{k=0}$

$$w_0 = \sqrt[4]{16}.e^{i\left(\frac{\pi}{4}+0.\frac{2\pi}{4}\right)} = 2e^{i\frac{\pi}{4}} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}(1+i)$$

$\underline{k=1}$

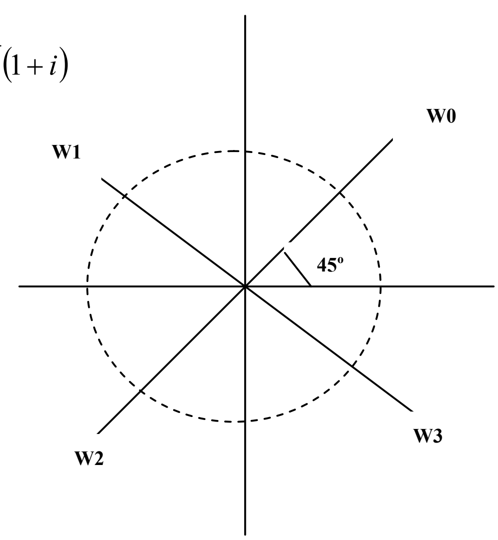
$$w_1 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}(-1+i)$$

$\underline{k=2}$

$$w_2 = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}(-1-i)$$

$\underline{k=3}$

$$w_3 = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2}(1-i)$$



H.W:

1. Express the answer below in the form z=re$^{i\theta}$.

$1.\frac{1+i}{1-i}, and,$
$2.(2+3i)(1-2i)$

$Answer:$

$1.e^{i\frac{\pi}{2}}, and, 2.\sqrt{65}e^{i\tan^{-1}(-0.125)}$

2. Find the three cube root of 1.

$Answer:$

$1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

3. Find the three cube roots of -8i.

$Answer:$

$2i, -\sqrt{3}-i, \sqrt{3}-i$

4. If $z=\frac{2+i}{1-i}$, find the real and imaginary parts of the complex number $z+\frac{1}{z}$.

5. If $z_1 = 2+i, z_2 = -2+4i, and, z_3 = \frac{1}{z_1}+\frac{1}{z_2}$, evaluate z_3 in the form *a+ib*, and draw z_1, z_2, and, z_3 in Argand diagram.

6. If $(2+3i)(3-4i) = x+iy$, evaluate x and y.

7. If $(a+b)+(a-b)i = (2+5i)^2 + i(2-3i)$, find the values of *a* and *b*.

8. When $z_1 = 2+3i, z_2 = 3-4i, and, z_3 = -5+12i$,then $z = z_1 + \frac{z_2 z_3}{z_2 + z_3}$. If *E=Iz*, find *E*, when *I=5+6i*.

9. If $\frac{R_1 + \omega L i}{R_3} = \frac{R_2}{R_4 - \frac{1}{\omega C}i}$,where, $R_1, R_2, R_3, R_4, \omega, L, and, C$ are real. Show that, $L = \frac{CR_2R_3}{\omega^2 C^2 R_4^2 + 1}$.

10. If $Z, and, \overline{Z}$ are conjugate complex numbers, find two complex numbers $Z = Z_1, and, Z = Z_2$,that satisfy the equation $3Z\overline{Z} + 2(Z - \overline{Z}) = 39 + 12i$.

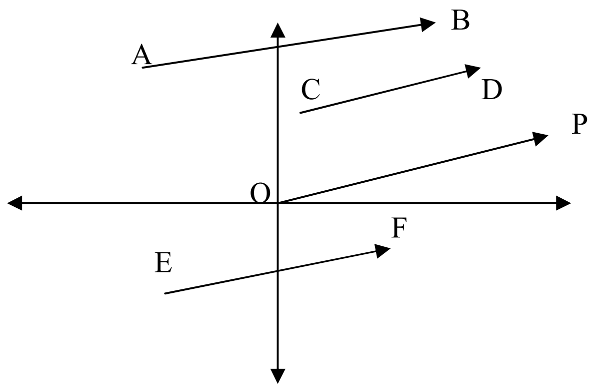
11. Find the values of x and y that satisfy the equation $(x+y)+(x-y)i = 14.8 + 6.2i$

Chapter Nine

Vectors

9.1 Vectors in the Plane:

A vector in the plane is a directed line segment. We call two vectors "equal" or "the same" if they have the same length and direction.

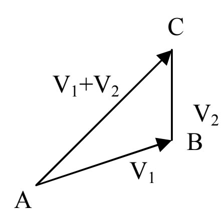


The four arrows have the same length and direction. They, therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.

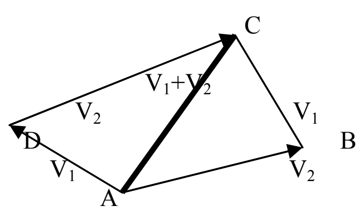
Geometric Addition: The Parallelogram Law

Two vectors V_1 and V_2 may be added geometrically

$$V_1 = \overrightarrow{AB}, and, V_2 = \overrightarrow{BC}$$
$$then, V_1 + V_2 = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



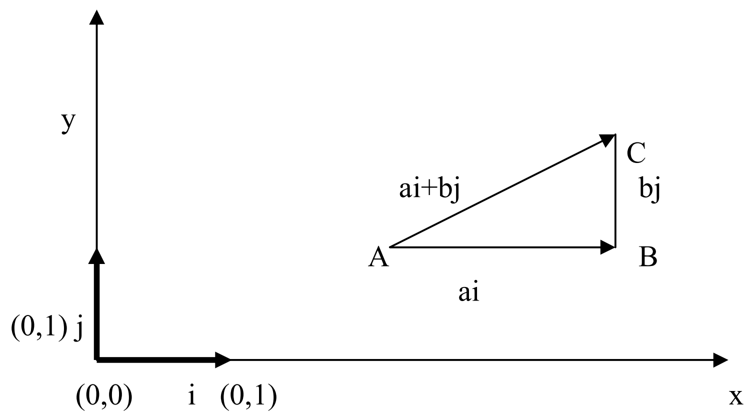
This addition is some times called the *parallelogram law* of addition because V_1+V_2 is given by diagonal of the parallelogram determined by V_1 and V_2.



Components:

A vector V can be written as the sum $V = V_1 + V_2$, the vectors V_1 and V_2 are said to be components of V.

The most common algebraic of vector is based on representation of each vector in terms of component parallel to the axes of Cartesian coordinate system.



- Along x-axis we choose vector i from (0,0) to (1,0) is *one basic vector.*
- Along y-axis choose vector j from (0,0) to (0,1) as *second basic vector.*
- Then ai and bj is scalar represents a vector parallel to x- and y-axis.

Then, the vector $V = \overrightarrow{AC}$ is the sum $V = ai + bj$. We call ai and bj the *vector components* of V in the direction of i and j. The numbers a and b are called the *scalar components* of V in the direction of i and j.

Definition:

Equality of vectors (Algebraic Definition)

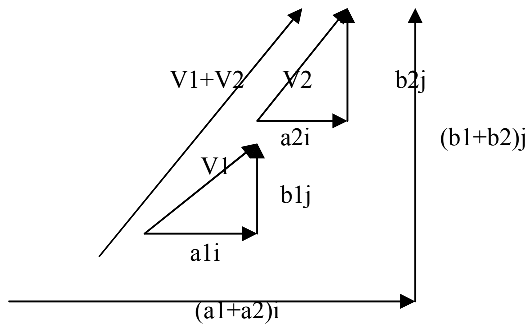
$$\boxed{ai + bj = a'i + b'j \Rightarrow a = a', and, b = b'}$$

That is, two vectors are equal if and only if their scalar components in the direction of i and j are equal.

Algebraic Addition:

Two vectors may be added algebraically by adding their scalar components

$$\boxed{\begin{array}{l} V_1 = a_1 i + b_1 j, and, V_2 = a_2 i + b_2 j \\ \therefore V_1 + V_2 = (a_1 + a_2)i + (b_1 + b_2)j \end{array}}$$



Example: $(2i-4j)+(5i+3j)=(2+5)i+(-4+3)j=7i-j$

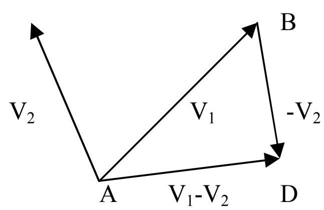
Subtraction:

The negative of vector (V) is the vector (-V) that has the same length but opposite direction. To subtract a vector V_2 from a vector V_1, we add $-V_2$ to V_1.

This may be done geometrically by two methods:

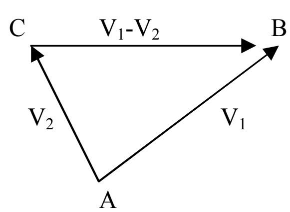
1. Drawing $-V_2$ from the tip of V_1 and then drawing the vector from initial point of V_1 to the tip of $-V_2$.

$$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$$
$$=V_1+(-V_2)$$
$$=V_1-V_2$$



2. Draw V_2 and V_1 both with a common initial point and then draw the vector from the tip of V_2 to the tip of V_1.

$$\overrightarrow{CB}=\overrightarrow{CA}+\overrightarrow{AB}$$
$$=-V_2+V_1$$
$$=V_1-V_2$$



In the term of component, algebraic subtraction:

$$\boxed{V_1-V_2=(a_1-a_2)i+(b_1-b_2)j}$$

Example: $(6i+2j)-(3i-5j)=(6-3)i+(2-(-5))j=3i+7j$

Length of a Vector:

Length of vector V=ai+bj is usually denoted by $|V|$:

$$\boxed{|V| = |ai + bj| = \sqrt{a^2 + b^2}}$$

Example: $|3i + 5j| = \sqrt{(3)^2 + (5)^2} = \sqrt{9 + 25} = \sqrt{34}$

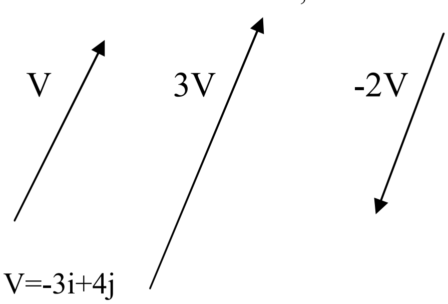
Multiplication by Scalars:

$$\boxed{c(ai + bj) = (ca)i + (cb)j}$$

Geometrically, cV is a vector whose length is $|c|$ times the length of V:

$$|cV| = |(ca)i + (cb)j| = \sqrt{(ca)^2 + (cb)^2} = |c|\sqrt{a^2 + b^2} = |c||V|$$

The direction of cV agree with that of V. If c=0, the vector cV has no direction.



Example: let c=2, and V=-3i+4j

$$|V| = |-3i + 4j| = \sqrt{(-3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

and

$$|2V| = |2(-3i + 4j)| = |-6i + 8j| = \sqrt{(-6)^2 + (8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10 = 2|V|$$

$if, c = -2$

$$|-2V| = |-2(-3i + 4j)| = |6i - 8j| = \sqrt{(6)^2 + (-8)^2} = \sqrt{100} = 10 = |-2||V|$$

Zero Vector:

The zero vector 0=0i+0j is called the zero vector. It is the vector whose length is zero, as we can see

$$|ai + bj| = \sqrt{a^2 + b^2} = 0 \Rightarrow a = b = 0$$

Unit Vector:

Any vector u whose length is equal to unit length used a long coordinate axes is called a *unit vector*.

$|i| = |1i + 0j| = \sqrt{1^2 + 0^2} = 1$

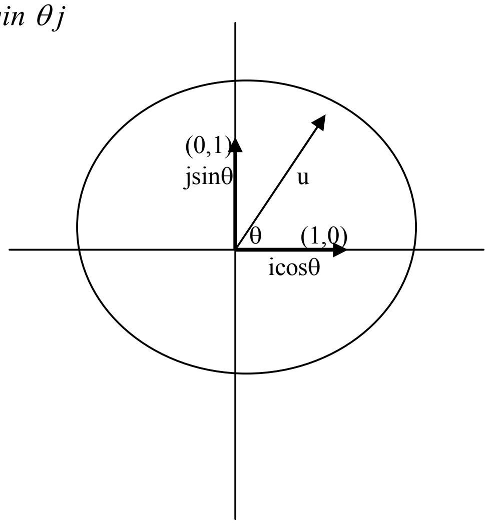
$|j| = |0i + 1j| = \sqrt{0^2 + 1^2} = 1$

If u is the unit vector obtained by rotating I through angle θ in the positive direction.

$u_x = \cos\theta$ (horizontal component)

$u_y = \sin\theta$ (vertical component)

So that, $u = \cos\theta\, i + \sin\theta\, j$



Direction:

The direction of a non zero vector obtained by dividing a vector A by its own length.

$$\boxed{Direction, of, A = \frac{A}{|A|}}$$

To see that $\frac{A}{|A|}$ is indeed a unit vector, we can calculate its length directly:

Length of $\frac{A}{|A|} = \left|\frac{A}{|A|}\right| = \left|\frac{1}{|A|}A\right| = \frac{1}{|A|}|A| = 1$

Note that, the zero vector has no direction.

Example: Find the direction of vector A=3i-4j

Solution:

$$Direction, of, A = \frac{A}{|A|} = \frac{3i-4j}{\sqrt{(3)^2+(-4)^2}} = \frac{3i-4j}{\sqrt{25}} = \frac{3}{5}i - \frac{4}{5}j$$

To check the length, we calculate:

$$\left|\frac{3}{5}i - \frac{4}{5}j\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}+\frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

Notes:

- It follow from the definition of direction that two non zero vectors A and B have the same direction, if and only if :

$$\boxed{\frac{A}{|A|}=\frac{B}{|B|}, or, A=\frac{|A|}{|B|}B}$$

- Also, if $A=kB$ k>0, then:

$$\frac{A}{|A|}=\frac{kB}{|kB|}=\frac{k}{|k|}\frac{B}{|B|}=\frac{k}{k}\frac{B}{|B|}=\frac{B}{|B|}$$

- We say that two non zero vectors A and B point in opposite direction if their directions are opposite in sign:

$$\boxed{\frac{A}{|A|}=-\frac{B}{|B|}}$$

Example: same direction

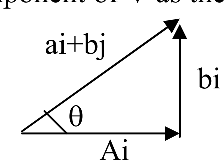
$A=3i-4j, and, B=\frac{3}{2}i-2j \Rightarrow \therefore B=\frac{1}{2}A \Rightarrow or, A=2B$

Example: opposite direction

$A=3i-4j, and, B=-9i+12j \Rightarrow \therefore B=-3A$

Slopes, Tangent, and Normal:

The slope of vector that is not parallel to y-axis. Thus, when a≠0, the vector V=ai+bj, has slope can be calculated from the component of V as the number b/a.



Example: Find the unit vectors tangent and normal to the curve $y=\frac{1}{2}x^3+\frac{1}{2}$ at the point (1,1).

Solution: slope of the line tangent to the curve at (1,1) is

$y'=\frac{3}{2}x^2=\frac{3}{2} \Rightarrow \therefore \frac{b}{a}=\frac{3}{2}$

$\therefore vector \Rightarrow V=2i+3j, has, slope=\frac{3}{2}$

$vector, length \Rightarrow |V|=\sqrt{2^2+3^2}=\sqrt{13}$

$\therefore unit, vector \Rightarrow u=\frac{V}{|V|}=\frac{2}{\sqrt{13}}i+\frac{3}{\sqrt{13}}j$

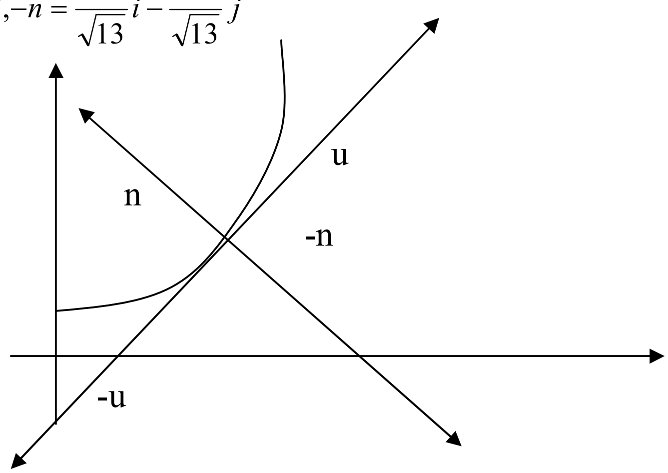
The vector u is tangent to the curve at (1,1) because it has the same direction of V.

$-u=-\frac{2}{\sqrt{13}}i-\frac{3}{\sqrt{13}}j$ is also tangent in opposite direction.

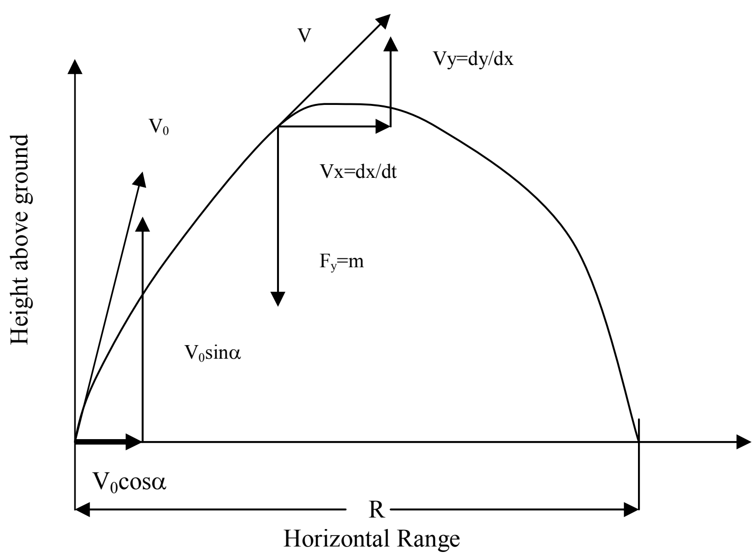
Slope of normal $=-\frac{a}{b}=-\frac{2}{3}$

$\therefore normal, n=-\frac{3}{\sqrt{13}}i+\frac{2}{\sqrt{13}}j$

$opposite, to, normal, -n=\frac{3}{\sqrt{13}}i-\frac{2}{\sqrt{13}}j$



9.2 Modeling Projectile Motion:



The position of the projectile t seconds after firing is:

$$x=(v_0\cos\alpha)t, and, y=(v_0\sin\alpha)t-\frac{1}{2}gt^2$$

Where:

v_0: initial speed.

α: angle of elevation or firing angle.

g: acceleration, 9.8 m/sec^2, or 32 ft/sec^2.

t: time.

Example: A projectile is fired over horizontal ground at an initial speed of 500 m/sec, at angle of elevation of 60^o. Where will the projectile be 10 seconds later?

Solution: v_o=500 m/s, α=60^o, and t=10 sec

$$x = (v_0 \cos\alpha)t = (500)\left(\frac{1}{2}\right)(10) = 2500m$$

$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 = (500)\left(\frac{\sqrt{3}}{2}\right)(10) - \frac{1}{2}(9.8)(10)^2 = 3840m$$

Then, after 10 sec., the projectile is 3840m in the air and 2500m down range.

Height, Range and Flight Time:

Example: A projectile is fired at an angle of elevation α and at initial speed v_o. When does the projectile reach its highest point? How high does the projectile rise?

Solution: the projectile reaches its heights point when its vertical velocity component is zero.

$$\frac{dy}{dt} = v_o \sin\alpha - gt = 0 \Rightarrow t = \frac{v_o \sin\alpha}{g}$$

For this value of t, the value of y is maximum:

$$y_{\max} = (v_o \sin\alpha)\left(\frac{v_o \sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_o \sin\alpha}{g}\right)^2$$

$$\boxed{\therefore y_{\max} = \frac{(v_o \sin\alpha)^2}{2g}}$$

Example: A projectile is fired over horizontal ground. When and where does it land?

Solution: To find when the projectile lands, we set y=0, and solve for t.

$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 = 0$$

$$t\left(v_0 \sin\alpha - \frac{1}{2}gt\right) = 0$$

$$\therefore either, t = 0, or, t = \frac{2v_0 \sin\alpha}{g}$$

Then, the projectile strikes the ground after $\boxed{t = \dfrac{2v_0 \sin\alpha}{g}}$

Horizontal range R is obtained by evaluating x at this value of t.

$$x = (v_o \cos\alpha)t$$

$$\therefore R = (v_o \cos\alpha)\left(\frac{2v_o \sin\alpha}{g}\right) = \frac{v_0^2}{g}\sin 2\alpha$$

$$\boxed{\therefore R = \frac{v_0^2}{g}\sin 2\alpha}$$

Example: What angle of elevation gives the maximum horizontal range?

Solution:

The value of $R = \dfrac{v_0^2}{g}\sin 2\alpha$, is greatest when;

$$\sin 2\alpha = 1 \Rightarrow 2\alpha = \sin^{-1} 1 \Rightarrow 2\alpha = 90^o \Rightarrow \alpha = 45^o$$

So, the maximum horizontal range achieved with:

$$\boxed{\alpha = 45^o}$$

H.W:

- A projectile is fired at speed of 840 m/s, at an angle of elevation of 60^o. How long will it take to get 21 km down range?

  Answer: 50 sec

- A projectile is fired over level ground with an initial speed of 500 m/s at an angle of elevation of 45^o.
  1. When and how far away will the projectile strike?
  2. How high over head will the projectile be when it is 5 km down range?
  3. What is the highest the projectile will go?

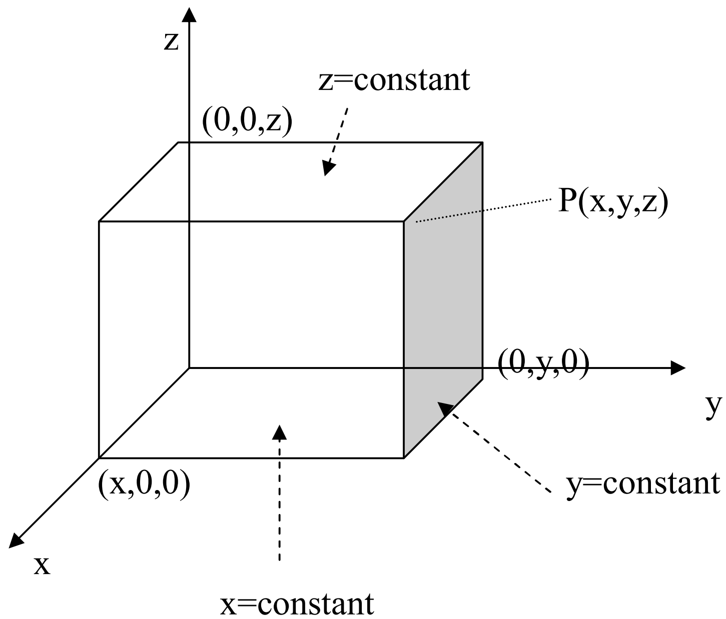
  Answer: 1. *72.2 sec, 25510.2 m,* 2. *4020m,* 3. *6377.6m.*

- Show that doubling the initial speed of projectile multiplies its maximum height and range by 4.
- A golf ball leaves the ground at 30^o angle at speed of 90 ft/sec. Will it clear the top of 30 ft tree 135 ft a way?

  Answer: No, y=29.9 ft

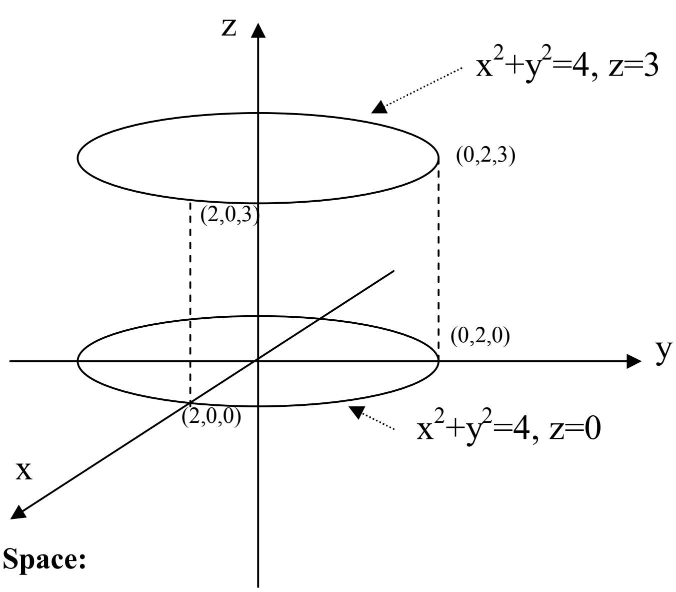
9.2 Coordinates and Vectors in Space:

Cartesian coordinates:



Example: identify the set of points P(x,y,z) whose coordinate satisfy the two equations $x^2+y^2=4$ $z=3$

Solution: the point lies in the horizontal plane z=3, and in this plane, make up the circle $x^2+y^2=4$.



Vector in Space:

The directed line segments we use to represent forces, displacements, and velocity in space are called vectors just as they are in the plane. The same rules of addition, subtraction and scalar multiplication are applied.

The vectors from the origin to the points (1,0,0), (0,1,0) and (0,0,1) are the basic vectors. We denote them by i, j, and k. The position vector R from the origin to the typical point P(x,y,z) is:

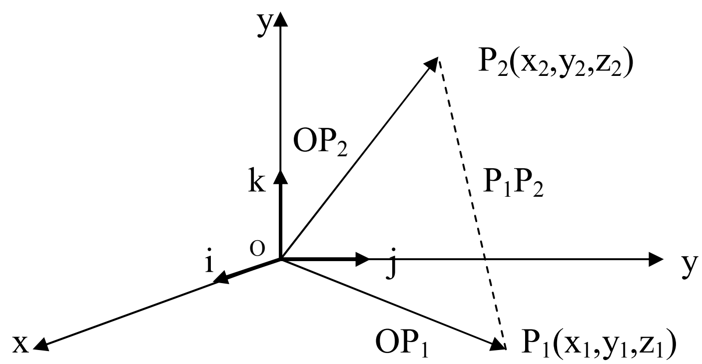
$$R = xi + yj + zk$$

The Vector between Two Points:

$$\overrightarrow{P_1P_2} = \overrightarrow{P_1O} + \overrightarrow{P_2O} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$
$$= x_2 i + y_2 j + z_2 k - x_1 i - y_1 j - z_1 k = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

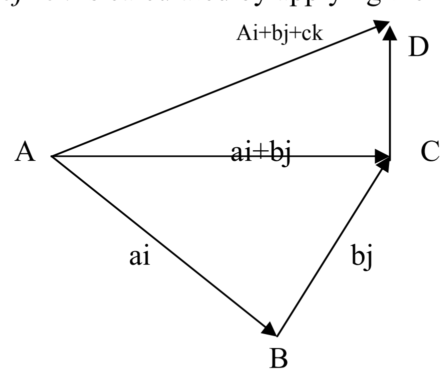
So, the vector from $P_1(x_1,y_1,z_1)$ to $P_2(x_2,y_2,z_2)$:

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k}$$



Length and Direction:

As in the plane, the important feature of a vector is the length and direction. The length of vector *ai+bj+ck* is calculated by applying the Pythagorean Theorem Twice.



$$\left|\overrightarrow{AC}\right| = |ai + bj| = \sqrt{a^2 + b^2}$$

$$|ai + bj + ck| = \left|\overrightarrow{AD}\right| = \sqrt{\left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{CD}\right|^2} = \sqrt{a^2 + b^2 + c^2}$$

The length of vector *A=ai+bj+ck*, is

$$\boxed{|A| = |ai + bj + ck| = \sqrt{a^2 + b^2 + c^2}}$$

Example: Find the length of A=i-2j+3k

Solution: $|A| = |i - 2j + 3k| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$.

The direction of a non zero vector A is the unit vector, obtained by dividing A by its length | A | .

$$\boxed{Direction, of, A = \frac{A}{|A|}}$$

Example: Find the direction of A=i-2j+3k

Solution:

The length is $|A| = |i - 2j + 3k| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$

$$Direction, of, A = \frac{A}{|A|} = \frac{i - 2j + 3k}{\sqrt{14}} = \frac{1}{\sqrt{14}}(i - 2j + 3k)$$

Example: Find a unit vector u in the direction of the vector form $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution:

$$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k = (3-1)i + (2-0)j + (0-1)k = 2i + 2j - k$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\therefore u = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{2i + 2j - k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

If we multiply a vector $A = a_1 i + a_2 j + a_3 k$, by a scalar c, the length of cA is | c | times the length of A. the reason is that:

$$cA = ca_1 i + ca_2 j + ca_3 k$$

$$So, |cA| = \sqrt{c^2 a_1^2 + c^2 a_2^2 + c^2 a_3^2} = |c|\sqrt{a_1^2 + a_2^2 + a_3^2} = |c||A|$$

Example: Find a vector 6 units long in the direction of A=2i+2j-k.

Solution:

$$6\frac{A}{|A|} = 6\frac{2i + 2j - k}{\sqrt{(2)^2 + (2)^2 + (-1)^2}} = 6\frac{2i + 2j - k}{3} = 4i + 4j - 2k$$

Distance in Space:

The distance between two point P_1 and P_2 in space is:

$$\boxed{\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

Example: Find the distance between $P_1(2,1,5)$ and $P_2(-2,3,0)$.

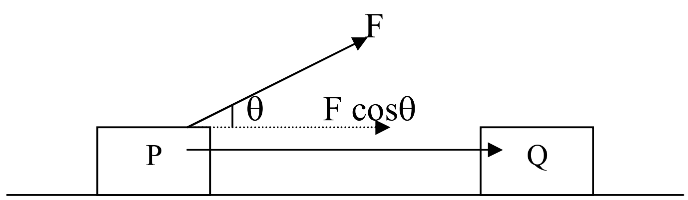
Solution:

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$= \sqrt{(-2-2)^2+(3-1)^2+(0-5)^2} = \sqrt{16+4+25} = \sqrt{45} = 3\sqrt{5}$$

9.3 Dot Product:

The dot product of two vectors, also called scalar products because the resulting products are numbers and not vectors.



The work done by constant force F during displacement $\overrightarrow{PQ}, is, \left(|F|\cos\theta\right)\left|\overrightarrow{PQ}\right|$.

Definition: Scalar or Dot Product A.B of two vector is:-

$$\boxed{A.B = |A||B|\cos\theta}$$

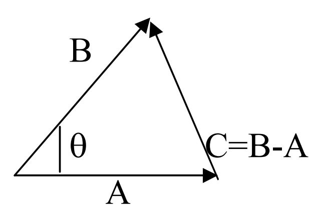
Where θ measures the smaller angle made by A and B.

Notes:

- In words, the scalar products of A and B is the length of A times the length of B times cosine the angle between A and B.
- Dot product of two vectors is positive when the angle between them is *acute*, and negative when the angle is *obtuse*.
- For the same vector A, θ=0, so that, $A.A = |A||A|\cos 0 = |A||A|(1) = |A|^2$.

Calculations: To calculate A.B from the components of A and B, we let

$$A = a_1 i + a_2 j + a_3 k$$
$$B = b_1 i + b_2 j + b_3 k$$
$$C = B - A = (b_1 - a_1)i + (b_2 - a_2)j + (b_3 - a_3)k$$
$$\boxed{A.B = a_1b_1 + a_2b_2 + a_3b_3}$$



Example: Find the angle θ between $A=i-2j-2k$ and, $B=6i+3j+2k$.

Solution:

$$A.B = a_1b_1 + a_2b_2 + a_3b_3 = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$
$$|A| = \sqrt{a^2 + b^2 + c^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$
$$|B| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{36+9+4} = \sqrt{49} = 7$$
$$\therefore A.B = |A||B|\cos\theta$$
$$\cos\theta = \frac{A.B}{|A||B|} = \frac{-4}{(3)(7)} = \frac{-4}{21} \Rightarrow \therefore \theta = \cos^{-1}\frac{-4}{21} = 101^o$$

Laws of Multiplication:

$$\boxed{\begin{array}{l} A.B = B.A \\ (cA).B = A.(cB) = c(A.B) \end{array}}$$

If $C=c_1i+c_2j+c_3k$, is any third vector then,

$$A.(B+C) = a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$$
$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) = A.B + A.C$$
$$\boxed{\therefore A.(B+C) = A.B + A.C}$$

And also,

$$\boxed{(A+B).(C+D) = A.C + A.D + B.C + B.D}$$

Orthogonal Vectors:

Two vectors whose scalar products is *zero* are said to be orthogonal. From the equation, $A.B = |A||B|\cos\theta$, we can see that neither $|A|, nor, |B|$ is zero, therefore, A.B is zero if and only if cosθ=0, that is when θ=90^o.

Example: Are the vector $A=3i-2j+k$ and $B=2j+4k$, orthogonal?

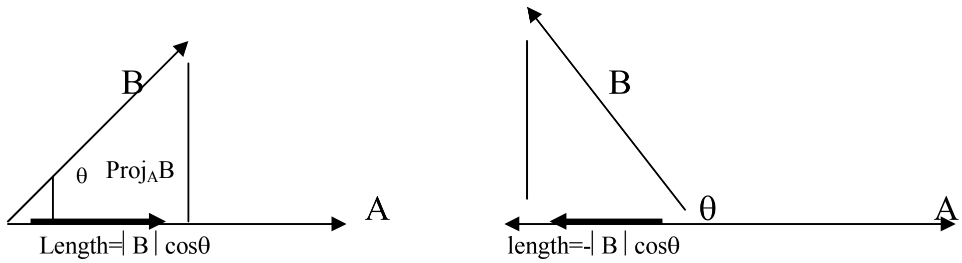
Solution:

$$A.B = (3)(0) + (-2)(2) + (1)(4) = -4 + 4 = 0$$

Yes, they are orthogonal.

Vectors Projection and Scalar Components:

The vector we get by projecting a vector B on to the line through a vector A is called the vector projection of B on to A.



If B represents a force, then the vector projection of B on to A represents the effective force in the direction of A.

- When θ is acute, length of vector projection of B on to A ($proj_A B$) is $|B|\cos\theta$.
- When θ is obtuse, length of vector projection of B on to A ($proj_A B$) is $-|B|\cos\theta$.
- The number $|B|\cos\theta$ is called *Scalar Component* of B in the direction of A.

$$\boxed{\begin{array}{l} A.B = |A||B|\cos\theta \Rightarrow (\div |A|) \\ \therefore |B|\cos\theta = \dfrac{A.B}{|A|} = B.\dfrac{A}{|A|} \end{array}}$$

A useful way to find $proj_A B$ is:

$$\boxed{proj_A B = \frac{A.B}{A.A} A} \qquad \text{Note that, } |A|^2 = A.A$$

Example: Find the vector projection of *B=6i+3j+2k*, on to *A=i-2j-2k*, and the scalar component of B in the direction of A.

Solution:

$$proj_A B = \frac{A.B}{A.A} A$$

$$= \frac{6-6-4}{1+4+4}(i-2j-2k) = -\frac{4}{9}(i-2j-2k) = -\frac{4}{9}i + \frac{8}{9}j + \frac{8}{9}k$$

The scalar component of B in the direction of A is

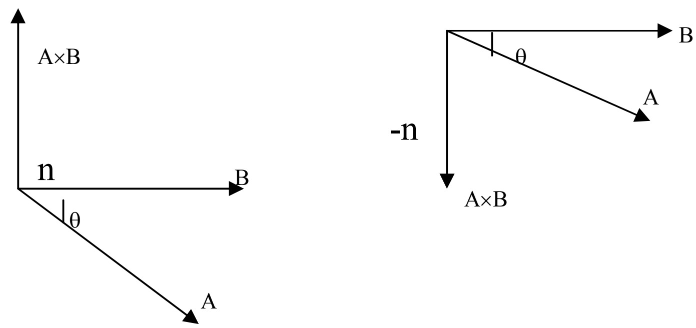
$$\therefore |B|\cos\theta = B.\frac{A}{|A|} = (6i+3j+2k).\left(\frac{1}{3}i - \frac{2}{3}j - \frac{2}{3}k\right) = \left(2-2-\frac{4}{3}\right) = -\frac{4}{3}$$

9.5 Cross Products:

$$\boxed{A \times B = n|A||B|\sin\theta}$$

- n is a unit vector.
- The vector product of A and B is often called the cross product of A and B.

- Dot product A.B is scalar, while A×B is a vector .
- If θ approaches 0^o or 180^o, the length of A×B approaches zero, therefore, A×B=0 if A and B are parallel.

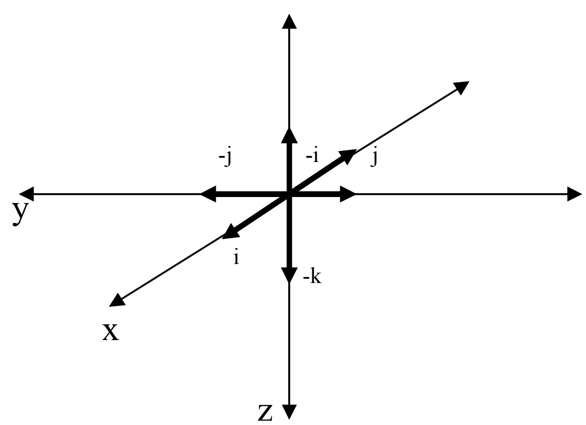


$$\boxed{B \times A = -(A \times B)}$$

Therefore, cross product is not commutative.

When we apply cross product to the unit vector i, j, and k.

$i \times j = -(j \times i) = k$
$j \times k = -(k \times j) = i$
$k \times i = -(i \times k) = j$
$i \times i = j \times j = k \times k = 0$



|A×B| is the Area of Parallelogram:

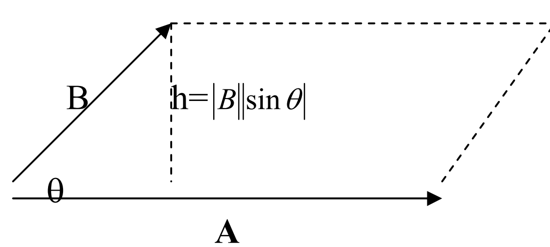
Because n is a unit vector, the magnitude of A×B is:

$$\boxed{|A \times B| = |n||A||B||\sin\theta| = |A||B||\sin\theta|}$$

This is the area of parallelogram determined by A and B.

$|A|$ is the base of parallelogram.

$|B||\sin\theta|$ is the height.



Vector Distribution Laws:

$$A\times(B+C)=A\times B+A\times C$$
$$and,$$
$$(B+C)\times A=B\times A+C\times A$$

The Determinant Formula for A×B:

We can use the following rule to calculate A×B from the component of A and B. If $A=a_1i+a_2j+a_3k$, and, $B=b_1i+b_2j+b_3k$, then

$$\boxed{A\times B=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

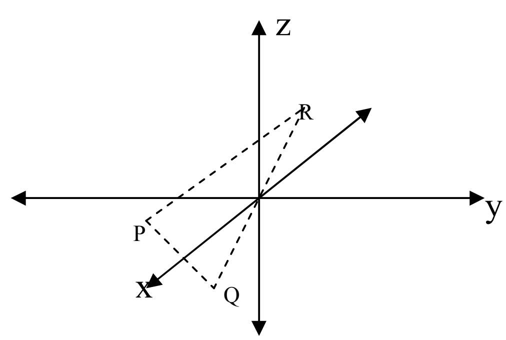
Example: Find A×B and B×A if, $A=2i+j+k$, and $B=-4i+3j+k$.

Solution:

$$A\times B=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}=\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}i-\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}j+\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}k=-2i-6j+10k$$
$$B\times A=-(A\times B)=2i+6j-10k$$

Example: Find the area of triangle whose vertices are P(1,-1,0), Q(2,1,-1) and R(-1,1,2).

Solution:



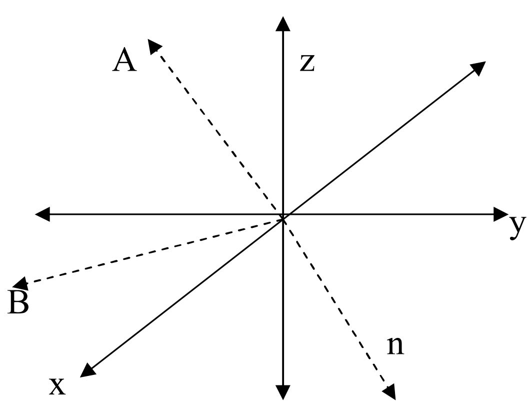
The area of triangle PQR=$\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|$

$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$
$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k = 6i + 6k$$
$$\therefore Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}|6i + 6k| = \frac{1}{2}\sqrt{72} = \frac{1}{2}\sqrt{(36)(2)} = 3\sqrt{2}$$

Example: Find a unit vector perpendicular to both $A=i+2k$ and $B=2i-2j$.

Solution:

$$A \times B = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4i + 4j - 2k$$
$$|A \times B| = \sqrt{16+16+4} = \sqrt{36} = 6$$
$$n = \frac{A \times B}{|A \times B|} = \frac{4i + 4j - 2k}{6} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$



H.W:

1. If $\overrightarrow{OA} = 4i + 3j, \overrightarrow{OB} = 6i - 2j, and \overrightarrow{OC} = 2i - j, find, \overrightarrow{AB}, \overrightarrow{BC}, and, \overrightarrow{CA}$, and deduce the length of the sides of the triangle ABC. $Ans. 2i - 5j, -4i + j, 2i + 4j; AB = \sqrt{29}, BC = \sqrt{17}, CA = \sqrt{20}.$
2. If $\vec{A} = 2i + 2j - k, and, \vec{B} = 3i - 6j + 2k, find, \vec{A}.\vec{B}, and, \vec{A} \times \vec{B}.$

   $Ans. -8, -2i - 7j - 18k$
3. Find the direction of the vector joining the two points (4,2,2) and (7,6,14). $Ans. (0.2308i + 0.3077j + 0.923k)$.

4. If $\vec{A}=2i+4j-3k, and, \vec{B}=i+3j+2k,$ determine the scalar and vector products, and the angle between the two given vectors. $Ans.\left(8,17i-7j+2k,\theta=66^{o}\right)$

5. If $\overrightarrow{OA}=2i+3j-k, and, \overrightarrow{OB}=i-2j+3k,$ determine, the value of $\overrightarrow{OA}.\overrightarrow{OB}$, and the product $\overrightarrow{OA}\times\overrightarrow{OB}$ in term of unit vectors, and the cosine of the angle between $\overrightarrow{OA}, and, \overrightarrow{OB}$. $Ans.\left(-7,7(i-j-k),\cos\theta=-0.5\right)$

6. Find the cosine of the angle between the vectors *2i+3j-k* and *3i-5j+2k*. *Ans.(cosθ=-0.4768)*

7. find the scalar product $\left(\vec{A}.\vec{B}\right)$, and the vector product $\vec{A}\times\vec{B}$, when:
$(1)\vec{A}=i+2j-k, \vec{B}=2i+3j+k, (2)\vec{A}=2i+3j+4k, \vec{B}=5i-2j+k$.
$Ans.(1)7,5i-3j-k,(2)8,11i+18j-19k$

8. Find the unit vector perpendicular to each of the vectors *2i-j+k*, and, *3i+4j-k*. Calculate the sine of the angle between the two vectors. $Ans.\left(-\frac{3}{\sqrt{155}}i+\frac{5}{\sqrt{155}}j+\frac{11}{\sqrt{155}}k,\sin\theta=0.997\right)$

9. If *A(1,-1,2), B(-1,2,2)* and *C(4,3,0)*, find the direction *BA* and *BC*, and hence show that the angle *ABC=69^{o}*. $Ans.\left(\frac{2i-3j+0k}{\sqrt{13}},\frac{5i+j-2k}{\sqrt{30}}\right)$

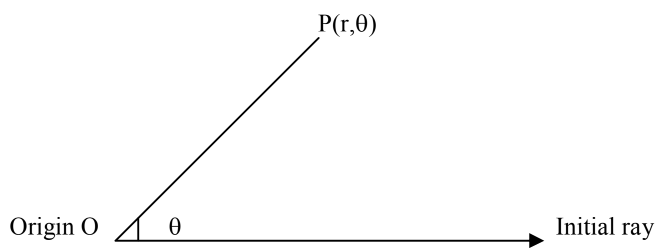
10. If $\vec{A}=3i-j+2k, and, \vec{B}=i+3j-2k,$ determine the magnitude and direction of the product vector $\vec{A}\times\vec{B}$, and show that it is perpendicular to the vector $\vec{C}=9i+2j+2k$. $Ans.\left(6\sqrt{5},\frac{-2i+4j+5k}{3\sqrt{5}}\right)$.

Chapter Ten

Polar Coordinates

10.1 The Polar Coordinates System:

- One of the distinctions between polar and Cartesian coordinates is that while a point in the plane has just one pair of Cartesian coordinates, it has many pairs of polar coordinates.
- To define polar coordinates we first fix an origin (O) and initial ray from (O). The each point P can be located by it polar coordinate pair (r, θ).

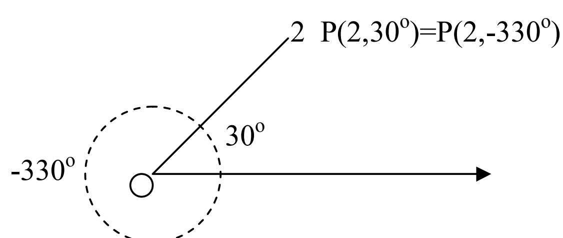


So, polar coordinate P(r,θ)

r: direct distance from O to P.

θ: direct angle from initial ray to OP.

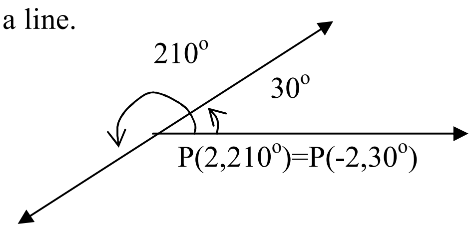
As in trigonometry, the angle θ is positive when measured counterclockwise and negative when measured clockwise. Note that angle of given point is not unique.



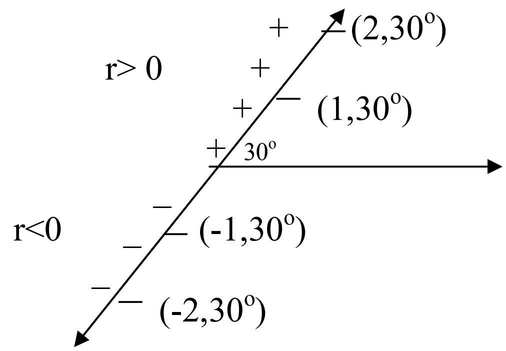
Negative Value of (r):

There are occasion when we wish to allow r to be negative. That is why we say "direct distance" from O to P.

The rays $\theta=30^o$ and $\theta=210^o$ make a line.

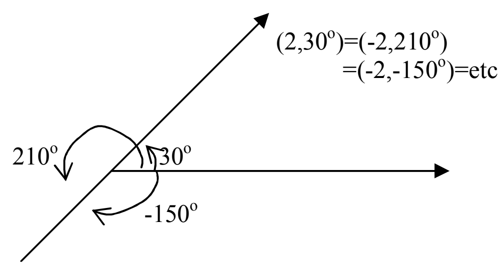


So, point $(2,210^o)$ can be reached by turning 210^o counterclockwise from initial ray and going *forward* 2 units. Or also can be reached by turning 30^o counterclockwise from initial ray and going *backward* two units.



The ray $\theta=30^o$ and its opposite.

Example: Find all polar coordinates of the point $(2,30^o)$. Express the angle in radians as well as degree.



To convert to radian measure:-

For r=2

$30^o+1\times360=390^o$	$30^o-1\times360=-330^o$
$30^o+2\times360=750^o$	$30^o-2\times360=-690^o$
$30^o+3\times360=1110^o$	$30^o-3\times360=-1050^o$

Then the polar coordinates *$(30^o+2n\times360^o)$* $\quad n = 0,\pm1,\pm2,......$

For r=-2

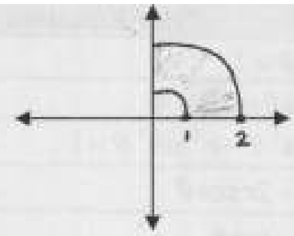
-150^o	-150^o
$-150^o+1\times360=210^o$	$-150^o-1\times360=-510^o$
$-150^o+2\times360=570^o$	$-150^o-2\times360=-870^o$
……..	…………

Then the polar coordinates *$(-2,-150^o+n\times360^o)$* $\quad n = 0,\pm1,\pm2,......$

Example: Graph the sets of points whose polar coordinates satisfy the following conditions:

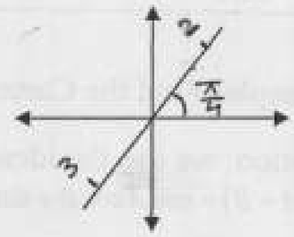
1.
$1 \le r \le 2$

$0 \le \theta \le \dfrac{\pi}{2}$

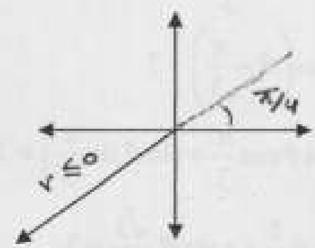


2.
$-3 \le r \le 2$

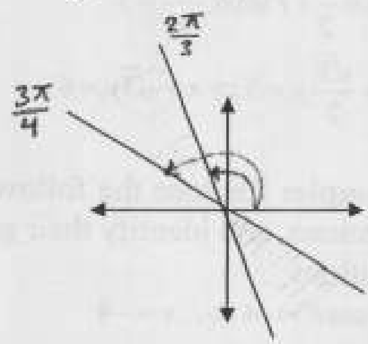
$\theta = \dfrac{\pi}{4}$



3.
$r \le 0 \Rightarrow \theta = \dfrac{\pi}{4}$



4.
$\dfrac{2\pi}{3} \le \theta \le \dfrac{3\pi}{4}$
(no restriction on r)



Cartesian vs. Polar Coordinates:

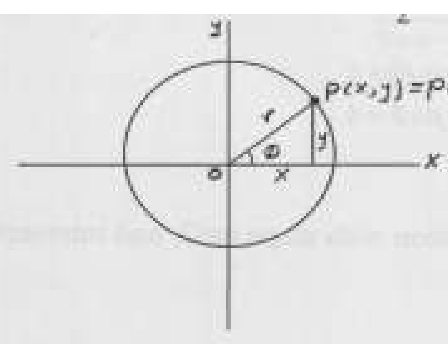
When we use both polar and Cartesian coordinates in a plane, we usually place the polar origin at the Cartesian origin and take initial ray of the polar coordinates to be the positive x-axis. The ray $\theta = \dfrac{\pi}{2}, r \ge 0$ is the nonnegative y-axis.

$x = r\cos\theta$

$y = r\sin\theta$

$\underline{or}, x^2 + y^2 = r^2$

$\tan\theta = \dfrac{y}{x}$



Examples:

Polar Equation	Cartesian Equation
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Example: Find the Cartesian equation for the curve $r\cos\left(\theta - \frac{\pi}{3}\right) = 3$

Solution: we use the identity

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

$$A = \theta, B = \frac{\pi}{3}$$

$$\therefore r\cos\left(\theta - \frac{\pi}{3}\right) = 3$$

$$r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 3$$

$$r\cos\theta.\frac{1}{2} + r\sin\theta.\frac{\sqrt{3}}{2} = 3$$

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3 \Rightarrow x + \sqrt{3}y = 6$$

Example: Replace the following polar equation by equivalent Cartesian equations, and identify their graphs.

Solution:

1. $r\cos\theta = -4 \Rightarrow \therefore x = -4$ Vertical line through x=-4 on x-axis.

2. $r^2 = 4r\cos\theta \Rightarrow \therefore x^2 + y^2 - 4x = 0$ Circle equation

3.

$$r = \frac{4}{2\cos\theta - \sin\theta}$$

$$r(2\cos\theta - \sin\theta) = 4$$

$$2r\cos\theta - r\sin\theta = 4$$

$$2x - y = 4$$

$$y = 2x - 4$$

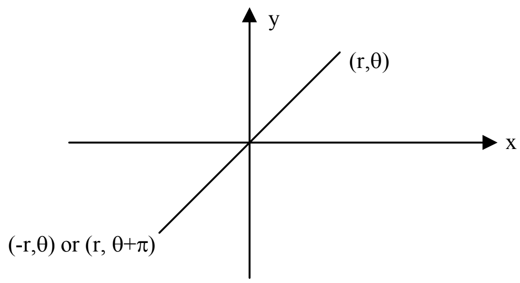
Line equation with slope m=2, and intercept b=-4.

10.2 Graphing in Polar Coordinates:

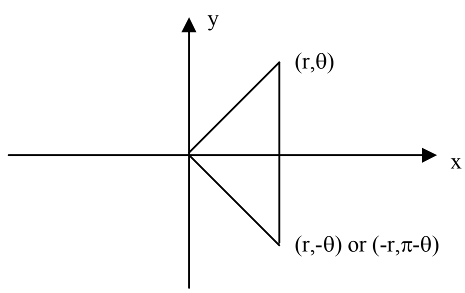
Symmetry and Slope at the origin

Three kinds of symmetry in graph of an equation f(r,θ)=0:

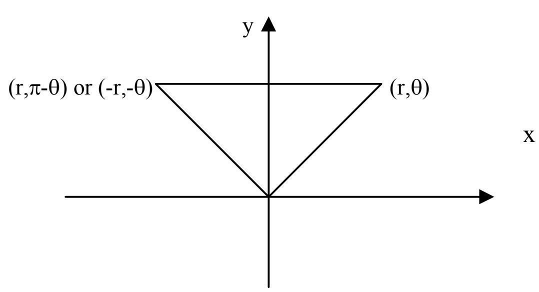
1. Symmetry about the origin if the equation is unchanged when r is replaced by –r, or when θ is replaced by θ+π.



2. Symmetry about x-axis if the equation is unchanged when θ is replaced by -θ, or the pair (r,θ) by the pair (-r,π-θ).



3. Symmetry about the y-axis if the equation is unchanged when θ is replaced by π-θ, or the pair (r,θ) by the pair (-r,-θ).



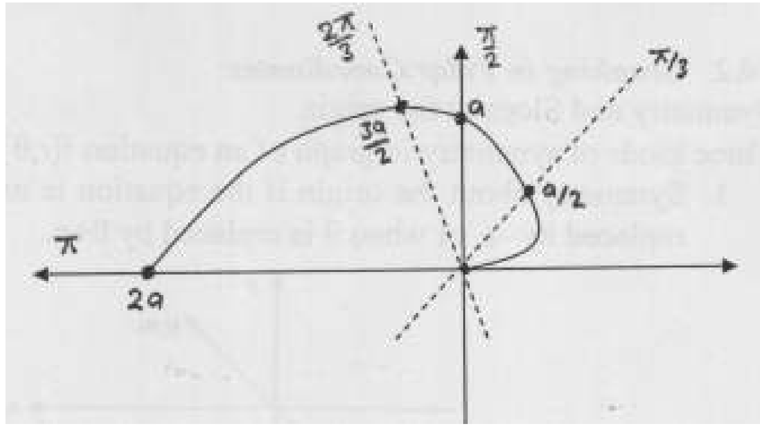
Example: Cardioids. Graph the curve $r = a(1-\cos\theta)$, a>0.

Solution: since $\cos(-\theta) = \cos\theta$, the equation is unchanged when θ is replaced by -θ, hence the curve is symmetry about x-axis. Also, since

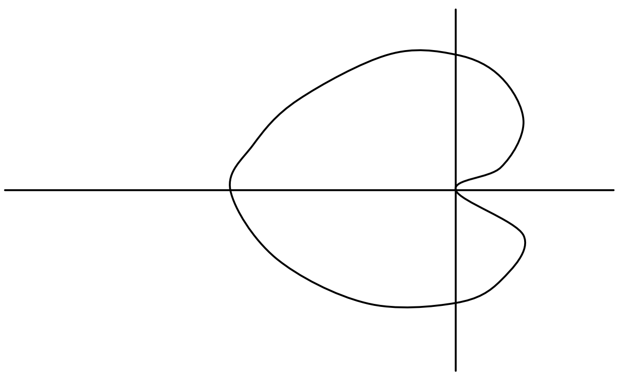
$$-1 \le \cos\theta \le 1$$
$$\therefore 0 \le r \le 2a$$
$$and, 0 \le \theta \le \pi$$

Then, make a table of *r vs.* θ, as follows:

θ	r
0	0
$\frac{\pi}{3}$	$\frac{a}{2}$
$\frac{\pi}{2}$	a
$\frac{2\pi}{3}$	$\frac{3a}{2}$
π	$2a$



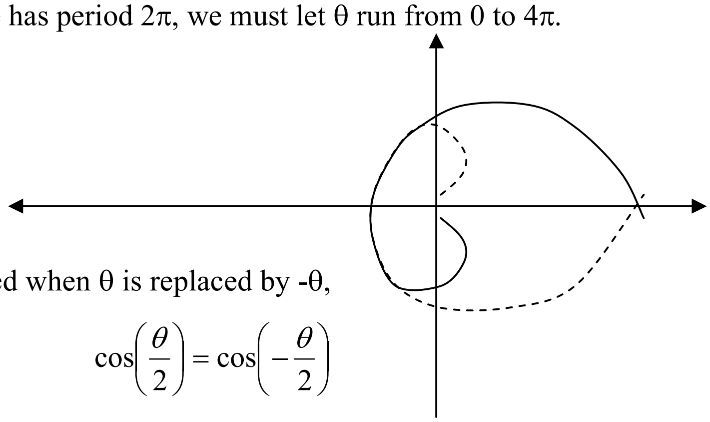
The graph is symmetry about x-axis, then the complete graph is called a cardioids because of its heart-shaped appearance.



Example: Graph the curve $r = 1 + \cos\frac{\theta}{2}$

Solution: since the cosine has period 2π, we must let θ run from 0 to 4π.

θ	r
0	2
π	1
2π	0
3π	1
4π	2



The equation is unchanged when θ is replaced by $-\theta$,

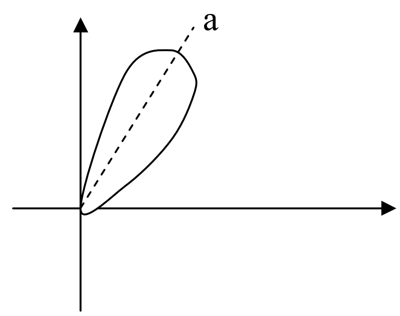
$$\cos\left(\frac{\theta}{2}\right) = \cos\left(-\frac{\theta}{2}\right)$$

Then, the curve is symmetry about x-axis.

Example: Graph the curve $r = a\sin 2\theta$

Solution: since, sine varies from $0 \to 2\pi$, So, we let θ varies from $0 \to \pi$

θ	r
0	0
$\frac{\pi}{4}$	$a\sin 2.\frac{\pi}{4} = a\sin\frac{\pi}{2} = a$
π	$a\sin 2\pi = 0$



Symmetry:

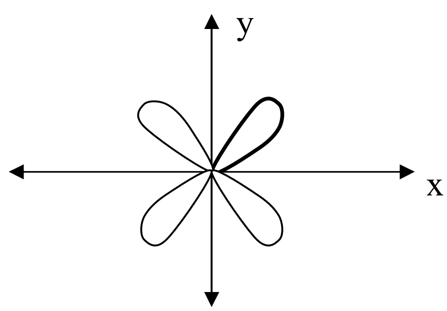
$r = a\sin 2\theta = 2a\sin\theta\cos\theta$

- Unchanged when θ replaced by -θ or (r,θ) by (-r,π-θ), because $\sin(\pi-\theta) = \sin\theta, and, \cos(\pi-\theta) = -\cos\theta$, $\therefore$ symmetry about x-axis.
- Unchanged when (r,θ) replaced by (-r,-θ), because $-r = a\sin(-2\theta) = -a\sin 2\theta$, $\therefore$ symmetry about y-axis.
- Unchanged when θ is replaced by (θ+π), because,

$$\sin(\theta+\pi) = \sin\theta\cos\pi + \sin\pi\cos\theta = -\sin\theta$$
$$\cos(\theta+\pi) = \cos\theta\cos\pi - \sin\theta\sin\pi = -\cos\theta$$

$\therefore$ symmetry about origin.

Then, the final figure is:

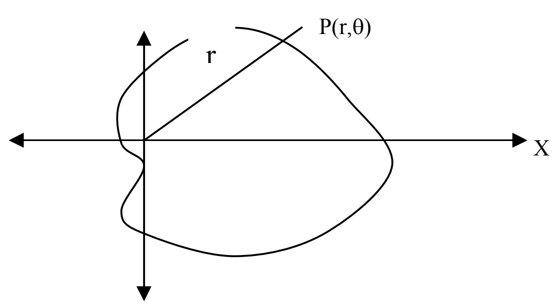


Area in the Plane:

Area between the origin and $r = f(\theta), where, \alpha \le \theta \le \beta$

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta}$$

Example: Find the area of the region enclosed by cardioids $r = 2(1+\cos\theta)$.



Solution:

$$A=\int_0^{2\pi}\frac{1}{2}r^2d\theta=\int_0^{2\pi}\frac{1}{2}.4.(1+\cos\theta)^2d\theta=2\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)d\theta$$

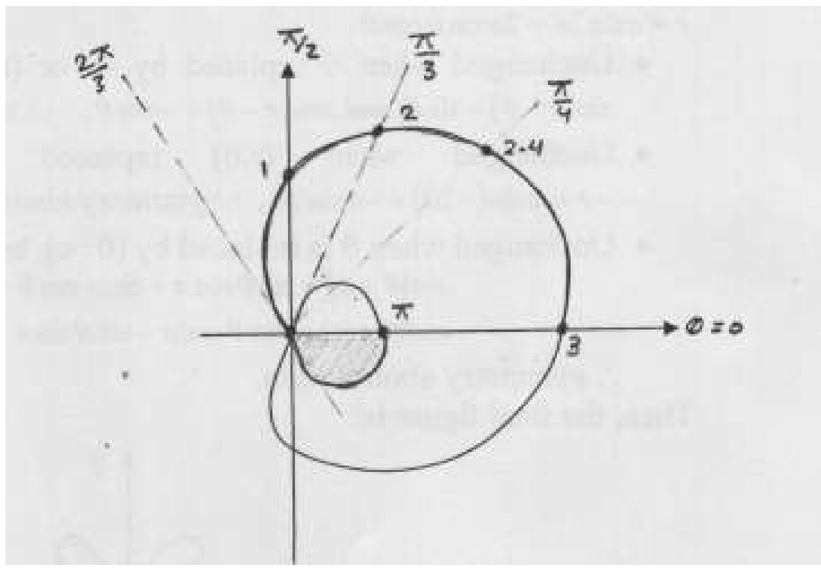
$$=\int_0^{2\pi}\left(2+4\cos\theta+2.\left(\frac{1+\cos 2\theta}{2}\right)\right)d\theta=\int_0^{2\pi}(3+4\cos\theta+\cos 2\theta)d\theta$$

$$=\left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi}=6\pi, unit, area$$

Example: Find the area inside the smaller loop of the limacon $r=2\cos\theta+1$.

Solution:

θ	r
0	3
$\frac{\pi}{4}$	2.4
$\frac{\pi}{3}$	2
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	0
π	-1



If we take half loop, we multiply the area by 2

$$A=2\int_{\frac{2\pi}{3}}^{\pi}\frac{1}{2}r^2d\theta=\int_{\frac{2\pi}{3}}^{\pi}(2\cos\theta+1)^2d\theta=\int_{\frac{2\pi}{3}}^{\pi}(4\cos^2\theta+4\cos\theta+1)d\theta$$

$$=\int_{\frac{2\pi}{3}}^{\pi}\left(4\frac{(1+\cos 2\theta)}{2}+4\cos\theta+1\right)d\theta=\int_{\frac{2\pi}{3}}^{\pi}(3+4\cos\theta+2\cos 2\theta)d\theta$$

$$=[3\theta+\sin 2\theta+4\sin\theta]_{\frac{2\pi}{3}}^{\pi}=(3\pi)-\left(2\pi-\frac{\sqrt{3}}{2}+4.\frac{\sqrt{3}}{2}\right)=\pi-\frac{3\sqrt{3}}{2}$$

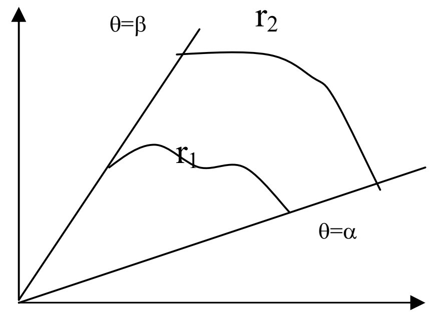
Or, we can the whole limits of integration without multiplication by 2, as follow:

$$A=\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\frac{1}{2}r^2d\theta=\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\frac{1}{2}(2\cos\theta+1)^2d\theta=.........etc$$

Area Between Curves:

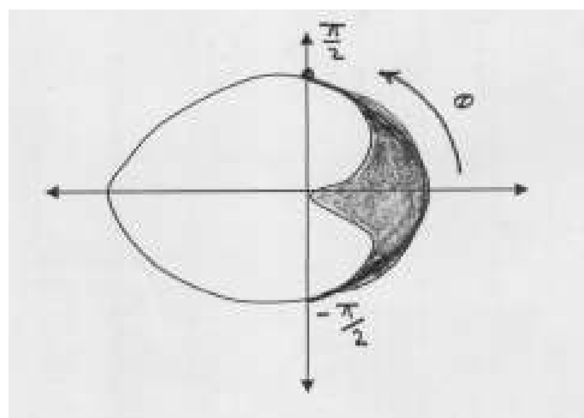
To find the area which lies between two polar curves from $\theta=\alpha$ to $\theta=\beta$, we subtract the integral of $\left(\frac{1}{2}r_1^2 d\theta\right)$ from the integral $\left(\frac{1}{2}r_2^2 d\theta\right)$:-

$$\boxed{\begin{array}{l} A=\int_{\alpha}^{\beta}\frac{1}{2}\left(r_2^2-r_1^2\right)d\theta \\ r_1\le r\le r_2 \\ \alpha\le\theta\le\beta \end{array}}$$



Example: Find the area of the region that lies inside the circle r=1 and outside the cardioids r=1-cosθ.

Solution: The outer curve $r_2=1$, the inner curve $r_1=1-\cos\theta$, and θ runs from $-\frac{\pi}{2}\to\frac{\pi}{2}$.



$$A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}\left(r_2^2-r_1^2\right)d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}\left(1^2-(1-\cos\theta)^2\right)d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-\left(1-2\cos\theta+\cos^2\theta\right)\right)d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2\cos\theta-\cos^2\theta\right)d\theta=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(2\cos\theta-\frac{(1+\cos 2\theta)}{2}\right)d\theta$$

$$=\frac{1}{2}\left[2\sin\theta-\frac{\sin 2\theta}{4}-\frac{\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{1}{2}\left(2-\frac{\pi}{4}\right)-\frac{1}{2}\left(-2+\frac{\pi}{4}\right)$$

$$=1-\frac{\pi}{8}+1-\frac{\pi}{8}=2-\frac{\pi}{4}$$